CHAPTER 6

Linear Operators and Functionals

In this chapter we discuss one of the central concepts of functional analysis — linear operators. We first establish the three most important results about general linear operators: the Banach–Steinhaus theorem, Banach's inverse mapping theorem, and the closed graph theorem. Next we proceed to considering linear functionals, i.e., operators with scalar values. The main results about linear functionals are connected with the Hahn–Banach theorem and its corollaries. A discussion of compact operators completes the main part of this chapter.

6.1. The Operator Norm and Continuity

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $A\colon X \to Y$ be a linear mapping, i.e., $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$. Linear mappings are also called *linear operators*. If $X = Y$, then the operator $I\colon x \mapsto x$ is called the *identity* or the *unit operator*. Linear mappings with values in $Y = \mathbb{R}$ or $Y = \mathbb{C}$ are called *linear functionals*. Set

$$\|A\| := \sup_{\|x\|_X \leqslant 1} \|Ax\|_Y$$

if this quantity is finite. We shall call $\|A\|$ the *operator norm* of A. For example, the norm of a linear functional l is defined by the equality

$$\|l\| := \sup_{\|x\|_X \leqslant 1} |l(x)|.$$

A linear operator with finite norm is called *bounded*. It should be noted that this terminology is not consistent with the terminology for the case of general mappings, when a mapping is called bounded provided that its image is bounded. For a bounded linear operator A only the image of the ball is bounded, but the image $\mathrm{Ran}\, A := A(X)$ of the whole space , called the *range of the operator* A, can be bounded only when $A(X) = 0$.

We observe that $\|A\|$ is the smallest number M such that $\|Ax\|_Y \leqslant M\|x\|_X$ for all $x \in X$. It is also clear that in the definition of $\|A\|$ we can take $\sup$ over the unit sphere in place of the unit ball (if $X \neq 0$).

If X, Y, Z are normed spaces, $A\colon Y \to Z$ and $B\colon X \to Y$ are bounded linear operators, then the linear operator $AB\colon X \to Z$ is obviously bounded and

$$\|AB\| \leqslant \|A\|\,\|B\|.$$

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This inequality can be strict; it is easy to construct an example in $\mathbb{R}^2$ or $\mathbb{C}^2$ (say, the composition of nonzero operators can be zero).

The set $\mathcal{L}(X,Y)$ of all bounded linear operators acting from a normed space X to a normed space Y is a normed space with the operator norm $A \mapsto \|A\|$. This space is obviously linear, since the algebraic sum of two bounded sets is bounded and the product of a bounded set by a scalar is bounded. It is proved below that a linear operator is bounded if and only if it is continuous (which is obviously false in both directions for nonlinear mappings).

The fact that the space of operators is normed follows at once from the relations $\|(A+B)x\|_Y \leqslant \|Ax\|_Y + \|Bx\|_Y$ and $\|\lambda Ax\|_Y = |\lambda|\,\|Ax\|_Y$. The main results of this chapter are connected with the operator norm. A particular role is played by the case where Y is the scalar field.

6.1.1. Definition. *Let X be a normed space. The space $X^* := \mathcal{L}(X,\mathbb{R})$ (or $X^* := \mathcal{L}(X,\mathbb{C})$ in the complex case) of all continuous linear functionals on the space X is called the dual (or topological dual) to the space X. The space X' of all linear functions on X is called the algebraic dual.*

The value of a linear functional f on a vector x is frequently denoted by $\langle f, x\rangle$. On concrete spaces it is easy to construct explicit examples of nonzero continuous functionals. It turns out that such functionals exist on every nonzero normed space. This highly non-obvious, but very important for the whole theory fact will be established below in §6.4 with the aid of the Hahn–Banach theorem. Algebraic duals are used relatively seldom.

6.1.2. Theorem. *For a linear operator $A\colon\ X \to Y$, the following conditions are equivalent*:

(i) *the operator A is bounded*;

(ii) *the operator A is continuous*;

(iii) *the operator A is continuous at some point.*

PROOF. If the operator A is bounded, then $\|Ax - Ay\| \leqslant \|A\|\|x-y\|$, i.e., the mapping A satisfies the Lipschitz condition with constant $\|A\|$ and hence is continuous. Suppose that the operator A is continuous at some point x_0. The equality $Ax = A(x-x_0) + Ax_0$ yields the continuity of A at the origin. Hence there exists $r > 0$ such that $\|Ax\| \leqslant 1$ whenever $\|x\| \leqslant r$. This gives the estimate $\|Ax\| \leqslant r^{-1}$ whenever $\|x\| \leqslant 1$. Thus, $\|A\| \leqslant r^{-1}$. □

6.1.3. Corollary. *A linear mapping between normed spaces is continuous precisely when it takes sequences converging to zero to bounded sequences.*

PROOF. The necessity of this condition is obvious, its sufficiency follows from the fact that if $\|x_n\| \leqslant 1$ and $\|Ax_n\| \to \infty$, then $y_n := \|Ax_n\|^{-1/2}x_n \to 0$ and $\|Ay_n\| = \|Ax_n\|^{1/2} \to \infty$. □

According to the established properties, a linear mapping discontinuous at one point is discontinuous everywhere. On a finite-dimensional normed space all linear operators are continuous, hence are bounded. In the infinite-dimensional case the situation is different.

6.1.4. Example. On every infinite-dimensional normed space there exists a discontinuous linear functional. Indeed, let $\{v_\alpha\}$ be a Hamel basis consisting of vectors of unit length. Let us pick in this basis a countable part $\{v_n\}$, set $l(v_n) = n$ for every n, on the remaining elements of the basis make l zero and extend by linearity to the whole space. It is clear that we have obtained an unbounded linear functional. It is discontinuous at every point by the previous theorem.

On some incomplete normed spaces one can construct explicitly (without using Hamel bases) discontinuous linear functionals. For example, on the space of continuous functions on $[0, 1]$ one can take the norm from $L^2[0, 1]$ and define l by the formula $l(x) = x(0)$. However, there are no explicit examples of unbounded linear functionals on Banach spaces.

Let us consider some examples of evaluation of norms of functionals and operators.

6.1.5. Example. (i) Let $X = C[0, 1]$ be equipped with the usual sup-norm and let $l(f) = f(0)$. Then $\|l\| = 1$, since $|l(f)| \leqslant 1$ if $\|f\| \leqslant 1$ and $l(1) = 1$.

(ii) Let $X = C[0, 1]$ be equipped with the usual sup-norm and let us set $l(f) = f(0) - f(1)$. Then $\|l\| = 2$, since $|l(f)| \leqslant 2$ if $\|f\| \leqslant 1$ and $l(f) = 2$ for the function $f\colon\ t \mapsto 1 - 2t$.

(iii) Let $X = C[0, 1]$ be equipped with the usual norm and let

$$l(f) = \int_0^1 f(t)g(t)\,dt, \tag{6.1.1}$$

where $g(t) = -1$ if $t \leqslant 1/2$, $g(t) = 1$ if $t > 1/2$. Then $\|l\| = 1$, since $|l(f)| \leqslant 1$ if $\|f\| \leqslant 1$, and for every $\varepsilon > 0$ there exists a continuous function f with $|f(t)| \leqslant 1$ and $l(f) > 1 - \varepsilon$. We observe that in this example there is no element f with $\|f\| \leqslant 1$ such that $l(f) = \|l\|$, i.e., a continuous linear functional can fail to attain its maximum on a closed ball.

More generally, for every integrable function g on $[0, 1]$ for the functional defined in (6.1.1) we have

$$\|l\| = \|g\|_{L^1} = \int_0^1 |g(t)|\,dt.$$

The bound $\|l\| \leqslant \|g\|_{L^1}$ is obvious, and for the proof of the equality it suffices to observe that for every $\varepsilon > 0$ there exists a step function g_ε on $[0, 1]$ such that $\|g - g_\varepsilon\|_{L^1} \leqslant \varepsilon$. Similarly to the previous case we observe that there exists a function $f_\varepsilon \in C[0, 1]$ with $\|f_\varepsilon\| \leqslant 1$ and

$$\int_0^1 f_\varepsilon(t)g_\varepsilon(t)\,dt \geqslant \|g_\varepsilon\|_{L^1} - \varepsilon.$$

This gives

$$l(f_\varepsilon) \geqslant \int_0^1 f_\varepsilon(t)g_\varepsilon(t)\,dt - \varepsilon \geqslant \|g\|_{L^1} - 3\varepsilon,$$

because we have $\|g_\varepsilon\|_{L^1} \geqslant \|g\|_{L^1} - \varepsilon$. Since ε was arbitrary, we obtain the opposite estimate $\|l\| \geqslant \|g\|_{L^1}$.

(iv) Let X be a Euclidean space and let $l(x) = (x, a)$, where $a \in X$. Then we have $\|l\| = \|a\|$, since $|l(x)| \leqslant \|x\|\|a\|$, which gives $\|l\| \leqslant \|a\|$. On the other hand, if $a \neq 0$, then $l(a/\|a\|) = \|a\|$.

(v) Let $X = L^2[0,1]$ be equipped with the usual norm and let

$$l(f) = \int_0^1 f(t)g(t)\,dt,$$

where $g \in L^2[0,1]$. Then $\|l\| = \|g\|_{L^2}$.

(vi) Let $X = C[0,1]$ and $Y = L^2[0,1]$ be equipped with their usual norms and let

$$Ax(t) = \left(\int_0^1 x(s)y(s)\,ds\right)\psi(t),$$

where $y \in L^1[0,1]$, $\psi \in L^2[0,1]$. Then $\|A\| = \|y\|_{L^1}\|\psi\|_{L^2}$. This follows from (iii) and the equality

$$\|Ax\| = \left|\int_0^1 x(s)y(s)\,ds\right| \|\psi\|.$$

(vii) A *diagonal operator* on a separable Hilbert space H is an operator of the form

$$Ax = \sum_{n=1}^{\infty} \alpha_n (x, e_n) e_n,$$

where $\{e_n\}$ is an orthonormal basis in H and $\{\alpha_n\}$ is a bounded sequence in $\mathbb{C}$. Then $\|A\| = \sup_n |\alpha_n|$, since $\|Ax\| \leqslant \sup_n |\alpha_n|$ and $\|Ae_n\| = |\alpha_n|$, whence one has $\|A\| \geqslant \sup_n |\alpha_n|$.

One should bear in mind that according to (iii) even in the case of a linear functional on an infinite-dimensional space the norm is not always attained on the unit ball, so sup cannot be always replaced by max.

We now prove an important result according to which any pointwise bounded family of continuous operators is uniformly bounded on the unit ball, i.e., is norm bounded.

6.1.6. Theorem. (THE BANACH–STEINHAUS THEOREM OR THE UNIFORM BOUNDEDNESS PRINCIPLE) *Suppose we are given a family $\{A_\alpha\}$ of bounded linear operators on a Banach space X with values in a normed space Y. Suppose that*

$$\sup_\alpha \|A_\alpha x\| < \infty \quad \textit{for every } x \in X.$$

Then $\sup\limits_\alpha \|A_\alpha\| < \infty$.

PROOF. As in Example 1.5.3 (we could refer to it at once), we consider the sets

$$M_n = \bigl\{x \in X\colon\ \|A_\alpha x\| \leqslant n \quad \text{for all } \alpha\bigr\}.$$

By the continuity of the operators A_α these sets are closed. By our hypothesis they cover X. According to Baire's theorem there exists n such that the set M_n contains some ball $U(a,r)$ centered at a with radius $r > 0$. Since $A_\alpha x = A_\alpha(x+a) - A_\alpha a$

and $\sup_\alpha \|A_\alpha a\| < \infty$, we obtain the uniform boundedness of the operators A_α on the ball $U(0, r)$, which gives their uniform boundedness on the unit ball. □

The completeness of X is essential in this theorem (although the requirement of completeness can be relaxed to the Baire property of X, which is obvious from the proof). For example, on the linear subspace in $C[0, 1]$ consisting of functions vanishing in a neighborhood of zero (its own neighborhood for every function), the bounded functionals $l_n(x) = nx(1/n)$ are pointwise bounded (at every fixed element x they are zero starting from some number), but their norms are not uniformly bounded: $\|l_n\| = n$. The same is true for the functionals $l_n(x) = nx_n$ on the linear subspace in l^2 consisting of all vectors with finitely many nonzero coordinates.

6.1.7. Corollary. *Let X and Y be Banach spaces and let $A_n\colon X \to Y$ be continuous linear operators such that for every vector x there exists a limit $Ax = \lim\limits_{n\to\infty} A_n x$ in Y. Then A is a continuous operator.*

PROOF. It is clear that A is a linear mapping. By the Banach–Steinhaus theorem we have $\sup_n \|A_n\| \leqslant C < \infty$. Then $\|Ax\| = \lim\limits_{n\to\infty} \|A_n x\| \leqslant C\|x\|$, hence $\|A\| \leqslant C$. □

6.1.8. Corollary. *In the situation of the previous corollary for every compact set $K \subset X$ we have*

$$\lim_{n\to\infty} \sup_{x\in K} \|Ax - A_n x\| = 0.$$

PROOF. We already know that there exists $C > 0$ such that $\|A_n\| \leqslant C$ for all n and that $\|A\| \leqslant C$. Let $\varepsilon > 0$. Let us take in K a finite $\varepsilon(4C)^{-1}$-net $x_1, \ldots, x_k$ and take N such that $\|Ax_i - A_n x_i\| \leqslant \varepsilon/2$ for $i = 1, \ldots, k$ and for all $n \geqslant N$. Then, for such n, for every $x \in K$ we have $\|Ax - A_n x\| \leqslant \varepsilon$, since there exists x_i with $\|x - x_i\| \leqslant \varepsilon(4C)^{-1}$, which by the triangle inequality gives

$$\begin{aligned}\|Ax - A_n x\| &\leqslant \|Ax - Ax_i\| + \|Ax_i - A_n x_i\| + \|A_n x_i - A_n x\| \\ &\leqslant 2C\|x - x_i\| + \varepsilon/2 \leqslant \varepsilon,\end{aligned}$$

as required. □

6.1.9. Theorem. *Let Y be a Banach space. Then for every normed space X the space of operators $\mathcal{L}(X, Y)$ is complete with respect to the operator norm.*

In particular, the space X^* is complete for every normed space X (not necessarily complete).

PROOF. Let $\{A_n\} \subset \mathcal{L}(X, Y)$ be a Cauchy sequence. For every $x \in X$ the sequence $\{A_n x\}$ is Cauchy in Y, since $\|A_n x - A_k x\| \leqslant \|A_n - A_k\|\|x\|$. Hence there exists $Ax = \lim\limits_{n\to\infty} A_n x$. It is clear that $A \in \mathcal{L}(X, Y)$, $\|A\| \leqslant \sup_n \|A_n\|$. However, we have to show that $\|A - A_n\| \to 0$. Let $\varepsilon > 0$. Let us find N such that $\|A_n - A_k\| \leqslant \varepsilon$ for all $n, k \geqslant N$. For all $n \geqslant N$ and each vector x of unit norm we have $\|Ax - A_n x\| = \lim\limits_{k\to\infty} \|A_k x - A_n x\| \leqslant \varepsilon$, so $\|A - A_n\| \leqslant \varepsilon$. □

Note that in this theorem and in the Banach–Steinhaus theorem we require the completeness of different spaces. For a beginner it is much easier to remember both theorems requiring the completeness of both spaces. However, then a moment reflexion shows that in the Banach–Steinhaus theorem the completeness of Y is not needed because one can pass to the completion of Y (which preserves the pointwise boundedness), while in the previous theorem we do not need the completeness of X, since all operators A_n can be extended to the completion of X giving a Cauchy sequence of operators on the completion.

The Banach–Steinhaus theorem can be also applied for obtaining negative results (which is sometimes called the "principle of condensation of singularities").

6.1.10. Example. Suppose that a sequence of continuous linear operators A_n from a Banach space X to a normed space Y is not norm bounded. Then there exists an element $x \in X$ such that $\sup_n \|A_n x\| = \infty$.

6.1.11. Example. For every $a \in [0, 2\pi]$ there exists a continuous 2π-periodic function for which the partial sums of the Fourier series at the point a are not uniformly bounded, in particular, have no finite limit.

PROOF. It suffices to consider $a = 0$. If for every function f in the space $C_{2\pi}$ of continuous functions f on $[0, 2\pi]$ with $f(0) = f(2\pi)$ the partial sums of the Fourier series at zero are bounded, then by virtue of representation (4.5.3) for the partial sums we have the pointwise boundedness of the sequence of functionals

$$l_n(f) := \int_0^{2\pi} f(t) \frac{\sin \frac{2n+1}{2} t}{2 \sin \frac{t}{2}}\, dt.$$

According to Example 6.1.5 the norm of this functional on $C_{2\pi}$ equals

$$\int_0^{2\pi} \left| \frac{\sin \frac{2n+1}{2} t}{2 \sin \frac{t}{2}} \right| dt.$$

Hence

$$\|l_n\| \geqslant \int_0^{2\pi} |\sin(n+1/2)t| \frac{1}{t}\, dt = \int_0^{(2n+1)\pi} |\sin s| \frac{1}{s}\, ds,$$

which tends to infinity as $n \to \infty$. □

In applications it is often useful to approximate in a suitable sense or replace infinite-dimensional operators by finite-dimensional ones. In particular, this is necessary for numeric methods. However, here there are many subtleties connected with the character of approximation. For example, the identity operator on an infinite-dimensional space cannot be approximated by finite-dimensional ones with respect to the operator norm. In §6.9 below we discuss compact operators, which on nice spaces (such as a Hilbert space) are approximated by finite-dimensional ones in the operator norm. But it is often sufficient to have some weaker approximations, for example, pointwise. This becomes possible for every bounded operator on a space with a Schauder basis; this question is discussed in §6.10(iv). In a Hilbert space with an orthonormal basis $\{e_n\}$, for every bounded operator A we have $P_n A x \to Ax$ for all x, where P_n is the orthogonal projection onto the linear span of $e_1, \ldots, e_n$, i.e., $P_n x = (x, e_1)e_1 + \cdots + (x, e_n)e_n$. Let us give a less trivial example.

6.1.12. Example. Let $C_{2\pi}$ be the space of all continuous 2π-periodic functions on the real line with the sup-norm and let σ_n be the operator taking a function f to its Fejér sum

$$\sigma_n(f)(x) := \int_0^{2\pi} f(x+z)\Phi_n(z)\,dz,$$

where Φ_n is the nth Fejér kernel (see (4.5.5)). Then by Theorem 4.5.9 we have $\|f - \sigma_n(f)\| \to 0$ for all $f \in C_{2\pi}$. Therefore, for every bounded operator A with values in $C_{2\pi}$, we obtain the pointwise convergence $\|Ax - \sigma_n \circ Ax\| \to 0$. A similar fact is true for the space $C[0,1]$. For this it suffices to observe that the space $C[0,\pi]$ can be embedded into $C_{2\pi}$ by the mapping $f \mapsto \widetilde{f}$, where $\widetilde{f}(t) = f(-t)$ for all $t \in [-\pi, 0]$, next $\widetilde{f}$ extends periodically.

We recall that according to Corollary 6.1.8 the pointwise convergence of operators on a Banach space yields the uniform convergence on compact sets, which increases the effect of such approximations.

6.2. The Closed Graph Theorem

The next result due to Banach and Schauder is fundamental for many other important results connected with operator ranges.

6.2.1. Lemma. *Let X and Y be Banach spaces with open unit balls U_X and U_Y and let $A\colon X \to Y$ be a continuous linear operator such that U_Y is contained in the closure of $A(U_X)$. Then $U_Y \subset A(U_X)$. In particular, $A(X) = Y$.*

PROOF. It follows from our assumption that

$$A(sU_X) \cap sU_Y \text{ is dense in } sU_Y \text{ for every } s > 0. \tag{6.2.1}$$

Let $y \in U_Y$ and $0 < \varepsilon < 1 - \|y\|$. Then $\|(1-\varepsilon)^{-1}y\| < 1$. Hence there is a vector $x_1 \in U_X$ for which $\|(1-\varepsilon)^{-1}y - Ax_1\| < \varepsilon$, i.e., $(1-\varepsilon)^{-1}y - Ax_1 \in \varepsilon U_Y$. By condition (6.2.1) there is a vector $x_2 \in \varepsilon U_X$ with $\|(1-\varepsilon)^{-1}y - Ax_1 - Ax_2\| < \varepsilon^2$. By induction with the aid of (6.2.1) we find $x_n \in \varepsilon^{n-1}U_X$ with

$$\|(1-\varepsilon)^{-1}y - Ax_1 - \ldots - Ax_n\| < \varepsilon^n.$$

Then $y = (1-\varepsilon)\sum_{n=1}^\infty Ax_n$. By the estimate $\|x_n\| < \varepsilon^{n-1}$ and the completeness of X the series $(1-\varepsilon)\sum_{n=1}^\infty x_n$ converges to some element $x \in X$. We have $Ax = y$ and $\|x\| < (1-\varepsilon)\sum_{n=1}^\infty \varepsilon^{n-1} = 1$, that is, $x \in U_X$. Thus, we have proved the inclusion $U_Y \subset A(U_X)$. □

6.2.2. Remark. We have used only the completeness of X. On the way we have obtained the following fact: if a set S is dense in U_Y, then every vector $y \in U_Y$ has the form $y = \sum_{n=1}^\infty c_n s_n$, where $s_n \in S$, $\sum_{n=1}^\infty |c_n| < 1$.

The next important theorem was obtained by Banach and Schauder.

6.2.3. Theorem. (THE OPEN MAPPING THEOREM) *Let X and Y be Banach spaces, $A \in \mathcal{L}(X, Y)$, and $A(X) = Y$. Then for every set V open in X the set $A(V)$ is open in Y.*

PROOF. Let U_X and U_Y be open unit balls in X and Y, respectively. Since $Y = \bigcup_{n=1}^{\infty} A(nU_X)$, by Baire's theorem there exists k such that the set $A(kU_X)$ is dense in some open ball $a + rU_Y$ of radius $r > 0$ in Y. Since we have $A(kU_X) = -A(kU_X)$, the set $A(kU_X)$ is dense in the ball $-a + rU_Y$. Hence $A(kU_X)$ is dense in the ball rU_Y. Indeed, if $\|y\|_Y \leqslant r$ and $u_n, v_n \in U_X$ are such that $A(ku_n) \to a + y$ and $A(kv_n) \to -a + y$, then $w_n := (u_n + v_n)/2 \in U_X$ and $A(kw_n) \to y$. Replacing A by $r^{-1}kA$, we can assume that $A(U_X)$ is dense in the ball U_Y. By the lemma proved above $U_Y \subset A(U_X)$. Therefore, we have $Ax + rU_Y \subset A(x + rU_X)$, $x \in X$, $r > 0$.

Suppose now that V is a nonempty open set in X. Let $y \in A(V)$, i.e., $y = Ax$, $x \in V$. Find $\varepsilon > 0$ such that $x + \varepsilon U_X \subset V$. Then

$$y + \varepsilon U_Y \subset A(x + \varepsilon U_X) \subset A(V).$$

Thus, $A(V)$ is open. For a generalization, see Exercise 12.5.31. □

6.2.4. Remark. (i) It is seen from the proof that in place of the surjectivity of A it suffices that $A(X)$ be a second category set in Y (then $A(U_X)$ will be dense in some ball in Y). In place of completeness of Y it suffices to have the Baire property for Y, but for arbitrary normed spaces Y the theorem is not valid: take the diagonal operator A on $X = l^2$ with eigenvalues n^{-1} and $Y = A(X)$ with the norm from l^2.

(ii) Since the image of the unit ball from X contains some ball $U_Y(0, \varepsilon)$ centered at the origin, we obtain that for every $y \in Y$ there exists a vector $x \in X$ such that

$$Ax = y \quad \text{and} \quad \|x\| \leqslant \varepsilon^{-1}\|y\|.$$

Of course, such a vector is not always unique.

An important corollary is the following result of Banach.

6.2.5. Theorem. (THE INVERSE MAPPING THEOREM) *Let A be a one-to-one continuous linear mapping of a Banach space X onto a Banach space Y. Then the inverse mapping A^{-1} is continuous.*

PROOF. The preimage under A^{-1} of an open set V in X coincides with $A(V)$ (because A is one-to-one) and is open in Y by the previous theorem. Hence A^{-1} is continuous. □

For nonlinear mappings A this theorem is false (Exercise 6.10.109).

6.2.6. Corollary. *Let X be a linear space that is complete with respect to two norms p_1 and p_2. Suppose that there exists a number c such that $p_1(x) \leqslant cp_2(x)$ for all $x \in X$. Then exists a number M such that $p_2(x) \leqslant Mp_1(x)$ for all $x \in X$.*

PROOF. By assumption the identity mapping of X with norm p_2 to X with norm p_1 is continuous. Hence the inverse mapping is continuous as well, i.e., it has finite norm, which means the existence of a desired number M. □

For formulating yet another important corollary of the open mapping theorem we introduce a new object.

The *graph of a mapping* $A\colon\ X \to Y$ is the set

$$\Gamma(A) := \{(x, Ax)\colon\ x \in X\} \subset X \times Y.$$

If X and Y are Banach spaces, then the product $X \times Y$ is equipped with the natural structure of a linear space and the natural norm $\|(x, y)\| := \|x\| + \|y\|$. It is clear that $X \times Y$ is complete with respect to this norm.

6.2.7. Theorem. (THE CLOSED GRAPH THEOREM) *A linear mapping between Banach spaces is continuous precisely when its graph is closed.*

PROOF. It is obvious that the graph of every continuous mapping is closed. The converse is false for nonlinear mappings. For a linear mapping $A\colon X \to Y$ with a closed graph we observe that this graph is a linear subspace in $X \times Y$ and hence is a Banach space. The operator $T\colon\ \Gamma(A) \to X$, $(x, Ax) \mapsto x$ is linear, continuous and maps $\Gamma(A)$ one-to-one onto X. By the inverse mapping theorem the operator $x \mapsto (x, Ax)$ is continuous. This yields the continuity of A. □

6.2.8. Corollary. *Let X, Y, Z be Banach spaces and let $j\colon\ Y \to Z$ be an injective continuous linear operator. Suppose that $A\colon\ X \to Y$ is a linear mapping such that the composition $j \circ A\colon\ X \to Z$ is continuous, i.e.,*

$$X \xrightarrow{A} Y \xrightarrow{j} Z$$

is a continuous mapping. Then A is continuous.

PROOF. We verify that the graph of A is closed. Let $x_n \to x$ in X and $Ax_n \to y$ in Y. It follows from our condition that

$$j(Ax_n) \to j(y), \quad j(Ax_n) = j \circ A(x_n) \to j \circ A(x).$$

Hence $j(y) = j \circ A(x)$, whence $y = Ax$. □

Let us give some typical examples of using the results obtained above.

6.2.9. Example. Let $A\colon\ L^2[a, b] \to L^2[a, b]$ be a continuous linear operator such that $A(L^2[a, b]) \subset C[a, b]$. Then the operator A is continuous as a mapping from $L^2[a, b]$ to $C[a, b]$.

6.2.10. Example. Suppose that a Banach space X is represented as a direct algebraic sum of its closed subspaces X_1 and X_2. Then the operators of algebraic projections $P_1\colon\ X \to X_1$ and $P_2\colon\ X \to X_2$ are continuous.

PROOF. Let us consider X as the Banach direct sum $X_1 \oplus X_2$ with the norm $(x_1, x_2) \mapsto \|x_1\| + \|x_2\|$. Since we always have $\|x_1 + x_2\| \leqslant \|x_1\| + \|x_2\|$, Corollary 6.2.6 shows that the new norm is equivalent to the original one. The projections are obviously continuous with respect to the new norm, hence they are continuous also with respect to the original one. □

One should bear in mind that the algebraic direct sum of two closed linear subspaces of a Banach space need not be closed (see Exercise 6.10.98). The previous example has the following interesting generalization.

6.2.11. Proposition. *Let X_1 and X_2 be closed subspaces of a Banach space X such that $X = X_1 + X_2$ (the sum is not supposed to be direct unlike the previous example). Then there exists a number $c > 0$ such that every element $x \in X$ admits a representation*

$$x = x_1 + x_2, \quad \textit{where } \|x_1\| + \|x_2\| \leqslant c\|x\|.$$

PROOF. Denote by Y the direct sum of the Banach spaces X_1 and X_2. The mapping $T\colon Y \to X$ defined by the formula $T(x_1, x_2) = x_1 + x_2$ is linear, continuous and surjective. The open mapping theorem yields that the image of the unit ball in Y contains a ball in X of some radius $r > 0$. Now it suffices to take the number $c := r^{-1}$. □

Let us prove one more useful result following from the already established facts. We observe that if X and Y are normed spaces, then any continuous linear operator $A\colon X \to Y$ with the *kernel* $\operatorname{Ker} A := A^{-1}(0)$ generates an injective linear operator

$$\widetilde{A}\colon X/\operatorname{Ker} A \to Y,$$

called the *factorization* of A *by its kernel* and defined by the formula $\widetilde{A}[x] := Ax$, where $[x]$ is the equivalence class in $X/\operatorname{Ker} A$ with a representative x. We have $\widetilde{A}(X/\operatorname{Ker} A) = A(X)$ and $\|\widetilde{A}\| = \|A\|$. The first equality is obvious, the second one us verified as follows: since $\|[x]\| \leqslant \|x\|$, we have $\|\widetilde{A}\| \geqslant \|A\|$. On the other hand, if $\|[x]\| = 1$ and $\varepsilon > 0$, then there exists a representative y of the equivalence class $[x]$ with $\|y\| \leqslant 1 + \varepsilon$, whence $\|\widetilde{A}[x]\| = \|Ay\| \leqslant (1 + \varepsilon)\|A\|$. Hence $\|\widetilde{A}\| \leqslant (1 + \varepsilon)\|A\|$ for all $\varepsilon > 0$ and so $\|\widetilde{A}\| \leqslant \|A\|$.

6.2.12. Proposition. *Let X and Y be Banach spaces and $A \in \mathcal{L}(X, Y)$. If the range of A has a finite codimension in Y, then it is closed.*

PROOF. Since $A(X) = \widetilde{A}(\widetilde{X})$, where $\widetilde{X} = X/\operatorname{Ker} A$ and $\widetilde{A}\colon \widetilde{X} \to Y$ is the operator generated by A, we can assume that the operator A is injective. By assumption there exists a finite-dimensional linear subspace Y_0 in Y such that Y is the algebraic direct sum of Y_0 and $A(X)$. Let $X \oplus Y_0$ denote the Banach direct sum of X and Y_0 (we recall that finite-dimensional normed spaces are complete). The operator $B\colon X \oplus Y_0 \to Y$, $(x, y) \mapsto Ax + y$ is continuous. This operator is injective, since we deal with the injective operator A. In addition, the operator B is surjective, since Y is the algebraic sum of $A(X)$ and Y_0. By the Banach theorem the operator B has a continuous inverse. Hence B takes closed sets to closed sets. In particular, the closed subspace X of the space $X \oplus Y_0$ (i.e., the set of pairs $(x, 0)$, $x \in X$) is taken to the closed set $A(X)$, which completes the proof. □

6.3. The Hahn–Banach Theorem

In this section we prove the most important result in linear analysis: the Hahn–Banach theorem. This result has numerous applications in mathematics as well as in applications, in particular, in economics. Unlike most of other assertions in this book, the Hahn–Banach theorem is nontrivial also in the finite-dimensional case.

6.3.1. Definition. *Let X be a linear space. A function $p\colon\ X \to [0,+\infty)$ is called a seminorm on X if for all scalars α and all vectors x, y we have*

$$p(\alpha x) = |\alpha| p(x) \quad \textit{and} \quad p(x+y) \leqslant p(x) + p(y).$$

6.3.2. Definition. *A function $p\colon\ X \to (-\infty,+\infty)$, where X is a real linear space, is called convex if*

$$p\bigl(tx + (1-t)y\bigr) \leqslant tp(x) + (1-t)p(y) \quad \forall\, x, y \in X,\ \forall\, t \in [0,1].$$

A function $p\colon\ X \to (-\infty,+\infty)$ is called positively homogeneous convex if

$$p(\alpha x) = \alpha p(x) \quad \textit{and} \quad p(x+y) \leqslant p(x) + p(y) \quad \forall\, \alpha \geqslant 0, \forall\, x, y \in X.$$

It is clear that all seminorms and all linear functions are positively homogeneous convex. It follows from the definition that a positively homogeneous convex function p is convex:

$$p(tx + (1-t)y) \leqslant p(tx) + p\bigl((1-t)y\bigr) = tp(x) + (1-t)p(y)$$

whenever $t \in [0,1]$. If also $p(-x) = p(x)$, then p is a seminorm.

6.3.3. Theorem. (THE HAHN–BANACH THEOREM) *Let X be a real linear space, p a convex function on X, X_0 a linear subspace in X, and l_0 a linear function on X_0 satisfying the condition*

$$l_0(x) \leqslant p(x) \quad \textit{for all } x \in X_0.$$

Then l_0 can be extended to a linear function l on all of X satisfying the condition $l(x) \leqslant p(x)$ for all $x \in X$.

PROOF. Note that this theorem is not trivial even in the two-dimensional case. As we shall now see, the main problem consists in extending the functional to a larger space in which X_0 is a hyperplane. Suppose first that X is the linear span of X_0 and a vector z not belonging to X_0. Every vector x in X has the form $x = x_0 + tz$. Every linear extension is uniquely determined by our choice of the number $c = l(z)$. Then $l(x) = l_0(x_0) + tc$. We have to pick c in such a way that $l \leqslant p$. Thus, we have to ensure the inequality

$$l_0(x_0) + tc \leqslant p(x_0 + tz). \tag{6.3.1}$$

For $t > 0$ this inequality is equivalent to the inequality

$$c \leqslant t^{-1} p(x_0 + tz) - l_0(t^{-1}x_0).$$

Similarly, for $t = -s < 0$ inequality (6.3.1) is equivalent to the estimate

$$c \geqslant -s^{-1} p(x_0 - sz) + l_0(s^{-1}x_0).$$

We show that there exists a number c satisfying both inequalities for all x_0 and t. To this end we prove the inequality

$$c' := \sup_{y \in X_0,\, s>0} \bigl[-s^{-1} p(sy - sz) + l_0(y)\bigr]$$

$$\leqslant c'' := \inf_{y \in X_0,\, t>0} \bigl[t^{-1} p(ty + tz) - l_0(y)\bigr]. \tag{6.3.2}$$

For c we then take any number between c' and c''. Inequality (6.3.2) is equivalent to the estimate

$$-s^{-1}p(sy' - sz) + l_0(y') \leqslant t^{-1}p(ty'' + tz) - l_0(y'') \quad \forall\, y', y'' \in X_0,\ s, t > 0,$$

which can be written as

$$l_0(y'') + l_0(y') \leqslant t^{-1}p(ty'' + tz) + s^{-1}p(sy' - sz).$$

Multiplying by the number $\lambda = ts/(t+s)$, for which $\lambda t^{-1} + \lambda s^{-1} = 1$, we write the latter as

$$l_0(\lambda y'' + \lambda y') \leqslant \lambda t^{-1}p(ty'' + tz) + \lambda s^{-1}p(sy' - sz).$$

This estimate is true, since

$$l_0(\lambda y'' + \lambda y') = l_0\bigl((\lambda t^{-1}(ty'' + tz) + s^{-1}(sy' - sz)\bigr)$$
$$\leqslant p\bigl((\lambda t^{-1}(ty'' + tz) + s^{-1}(sy' - sz)\bigr) \leqslant \lambda t^{-1}p(ty'' + tz) + \lambda s^{-1}p(sy' - sz).$$

Thus, in the considered case the theorem is proved. In the general case an extension is constructed step-by-step by adding an independent vector, which is done with the aid of Zorn's lemma. Let $\mathfrak{M}$ denote the collection of all possible extensions of l_0 to larger subspaces satisfying the condition of domination by p. Every such extension l' has a linear domain of definition L', on which $l' \leqslant p$, and $l'|_{X_0} = l_0$. We declare an extension l' subordinated to an extension l'' if for the corresponding domains of definition we have $L' \subset L''$ and $l''|_{L'} = l'$. It is clear that we obtain a partial order. The chain condition is fulfilled: if we are given a chain of extensions l_α with domains L_α, then a majorant $l \in \mathfrak{M}$ for it can be constructed as follows. The union L of all L_α is a linear space, since for every $x, y \in L$ there exist L_α and L_β with $x \in L_\alpha$ and $y \in L_\beta$, but by the definition of a chain either $L_\alpha \subset L_\beta$ or $L_\beta \subset L_\alpha$, i.e., in any case $x + y \in L$. It is clear that $tx \in L$ for all scalars t. By the same reasoning the function $l(x) = l_\alpha(x_\alpha)$ for $x = x_\alpha \in L_\alpha$ is well-defined on L, i.e., $l_\alpha(x_\alpha) = l_\beta(x_\beta)$ if $x_\alpha = x_\beta$. Moreover, $l \leqslant p$ on L. Thus, $l \in \mathfrak{M}$ is a majorant for all l_α. By Zorn's lemma $\mathfrak{M}$ contains a maximal element l. According to the first step, the domain of definition of l coincides with the whole space X: otherwise l could be linearly extended to a larger subspace with subordination to p contrary to the maximality of l. □

With the aid of a Hamel basis (see Proposition 1.1.1) it is easy to find a *linear* extension of l, but it is not always subordinated to p.

Usually various versions of the Hahn–Banach theorem are established for seminorms or positively homogeneous functions p (which is sufficient for most of applications), but the case of convex functions was also considered in the literature, see Altman [**665**], Bittner [**672**], it can be found in books, see, e.g., Barbu, Precupanu [**47**]. Mazur and Orlicz [**697**] initiated a study of more general extension problems for linear functionals and linear operators with several restrictions, there is an extensive literature on this direction.

6.3.4. Corollary. *Let X be a real or complex linear space, p a seminorm on the space X, X_0 a linear subspace in X, and let l_0 be a linear function on X_0*

satisfying the following condition:

$$|l_0(x)| \leqslant p(x) \quad \forall\, x \in X_0.$$

Then l_0 *can be extended to a linear function* l *on all of* X *satisfying the condition* $|l(x)| \leqslant p(x)$ *for all* $x \in X$.

PROOF. In the real case this assertion follows directly from the Hahn–Banach theorem. In the complex case, let $X_{\mathbb{R}}$ denote the realification of X, i.e., X over the field $\mathbb{R}$. Let us apply the Hahn–Banach theorem to the function $\mathrm{Re}\, l_0$ on the realification $X_{0,\mathbb{R}}$ of the space X_0. It is clear that $|\mathrm{Re}\, l_0| \leqslant p$ on $X_{0,\mathbb{R}}$. We obtain a real linear function l_1 on $X_{\mathbb{R}}$ with $l_1|_{X_{0,\mathbb{R}}} = \mathrm{Re}\, l_0$ and $l_1 \leqslant p$. We observe that $|l_1| \leqslant p$. We now set

$$l(x) = l_1(x) - il_1(ix), \quad x \in X,$$

which is possible, since $ix \in X$ and X coincides with $X_{\mathbb{R}}$ as a set. For all $x \in X_0$ we have

$$l_1(x) - il_1(ix) = \mathrm{Re}\, l_0(x) - i\mathrm{Re}\, l_0(ix) = \mathrm{Re}\, l_0(x) + i\mathrm{Im}\, l_0(x) = l_0(x).$$

Finally, for every $x \in X$ there exists a real number θ such that $l(x) = e^{i\theta}|l(x)|$. Set $y = e^{-i\theta}x$. Then $l(y) = |l(x)|$, i.e., $l(y) = l_1(y)$ and $l_1(iy) = 0$, since $l_1(iy), l(y) \in \mathbb{R}$. Hence $|l(x)| = l(y) = l_1(y) \leqslant p(y) = p(x)$. □

6.3.5. Corollary. *Let* X_0 *be a linear subspace of a normed space* X (*not necessarily closed*) *and let* l_0 *be a continuous linear function on* X_0. *Then* l_0 *can be extended to a continuous linear function on all of* X *with the same norm as the functional* l_0 *on* X_0.

PROOF. By assumption $|l_0(x)| \leqslant \|l_0\|\|x\|$ if $x \in X_0$. Set $p(x) = \|l_0\|\|x\|$. Applying the previous corollary, we extend l_0 to a linear function l on X with the bound $|l| \leqslant p$. This gives $\|l\| \leqslant \|l_0\|$. Since $\|l_0\| \leqslant \|l\|$, one has $\|l\| = \|l_0\|$. □

Let us give a geometric form of the Hahn–Banach theorem connected with separation of convex sets.

Let X be a real linear space. We shall say that a linear function l *separates* two sets $A, B \subset X$ if

$$\inf_{x\in A} l(x) \geqslant \sup_{x\in B} l(x).$$

In other words, there exists a number c such that

$$B \subset \{x\colon\ l(x) \leqslant c\} \quad \text{and} \quad A \subset \{x\colon\ l(x) \geqslant c\}.$$

Geometrically this means that A and B are on different sides from the affine subspace $l^{-1}(c)$.

If $l \neq 0$, then the set $\{x\colon\ l(x) \leqslant c\}$ is called a *halfspace* and the set $\{x\colon\ l(x) = c\}$ is called a *hyperplane*.

The *algebraic kernel* of a set A in a linear space X is defined as the set of all points $x \in A$ such that for every $v \in X$ there exists a number $\varepsilon = \varepsilon(v) > 0$ for which $x + tv \in A$ whenever $|t| < \varepsilon$. If X is a normed space, then every inner point of A belongs to the algebraic kernel, but the algebraic kernel can be larger than the interior. For example, if we take for X the space of polynomials on $[0,1]$ with the norm from $C[0,1]$, then the set of polynomials x with $\max_t |x'(t)| < 1$ has no interior, but coincides with its algebraic kernel. For an arbitrary infinite-dimensional normed space X, the set $l^{-1}\{(-1,1)\}$, where l is a discontinuous linear function, also has no inner points, but coincides with its algebraic kernel.

Let V be a convex set in a linear space X. Suppose that the algebraic kernel of V contains the point 0.

The *Minkowski functional* of the set V is the function

$$p_V(x) := \inf\{t > 0\colon\ t^{-1}x \in V\}.$$

The condition that 0 belongs to the algebraic kernel of V is needed to guarantee that the functional p_V be with finite values.

6.3.6. Theorem. *Under the stated assumptions, the functional p_V is positively homogeneous convex and nonnegative. If the set V is balanced (i.e., $\theta V \subset V$ whenever $|\theta| \leqslant 1$), then p_V is a seminorm.*

Conversely, for every positively homogeneous convex nonnegative function p the set $U := \{x\colon\ p(x) \leqslant 1\}$ is convex, its algebraic kernel is the set $\{x\colon\ p(x) < 1\}$, and $p = p_U$.

PROOF. As we have already noted, $0 \leqslant p_V(x) < \infty$. For every $\alpha > 0$ and every $x \in X$ we have

$$p_V(\alpha x) = \inf\{t > 0\colon\ t^{-1}\alpha x \in V\} = \alpha \inf\{s > 0\colon\ s^{-1}x \in V\} = \alpha p_V(x).$$

Let $x, y \in X$. Let us fix $\varepsilon > 0$ and choose $s, t > 0$ such that

$$p_V(x) < s < p_V(x) + \varepsilon, \quad p_V(y) < t < p_V(y) + \varepsilon.$$

Then $x/s \in V$, $y/t \in V$. Set $r = s + t$. The point $(x+y)/r = \frac{s}{r}x/s + \frac{t}{r}y/t$ belongs to the interval with the endpoints x/s and y/t, hence by the convexity of the set V belongs to V. Thus, $p_V\bigl((x+y)/r\bigr) \leqslant 1$, whence

$$p(x+y) \leqslant r < p_V(x) + p_V(y) + 2\varepsilon.$$

Since ε was arbitrary, we obtain $p_V(x+y) \leqslant p_V(x) + p_V(y)$. Finally, if V is balanced, then $p_V(\theta x) = p_V(x)$ whenever $|\theta| = 1$.

Let $p \geqslant 0$ be a positively homogeneous convex function. By the convexity of p the set $U = \{x\colon\ p(x) \leqslant 1\}$ is convex. Every element $x \in X$ with $p(x) < 1$ belongs to the algebraic kernel U, because

$$p(x+ty) \leqslant 1 \quad \text{whenever } |t| < (1-p(x))/\max(p(y),1).$$

If $p(x) \geqslant 1$, then x does not belong to the algebraic kernel of U, since

$$p(x+\varepsilon x) = (1+\varepsilon)p(x) \geqslant 1+\varepsilon > 1$$

for all $\varepsilon > 0$. It is clear from the definition that $p_U = p$. □

6.3.7. Theorem. *Let U and V be convex sets in a real linear space X such that the algebraic kernel of U is not empty and does not intersect V. Then there exists a nonzero linear function separating U and V.*

PROOF. We can assume that 0 belongs to the algebraic kernel of U (otherwise we can pass to the sets $U - a$ and $V - a$ for some point a in the algebraic kernel of U). Let us take a point $v_0 \in V$ and set $W = U - V + v_0$. It is straightforward to verify that W is a convex set and its algebraic kernel W^0 contains 0. It is easy to derive from the definition that $\lambda u \in U^0$ and $\lambda w \in W^0$ for all $u \in U^0$, $w \in W^0$, $|\lambda| < 1$. Finally, we observe that $v_0 \notin W^0$. Otherwise the algebraic kernel of $U - V$ contains 0. Then there exists a nonzero element $x \in U \cap V$ and $tx \in U - V$ for some $t \in (0,1)$, i.e., $tx = u - v$, $u \in U$, $v \in V$ and $(t+1)^{-1}u = t(t+1)^{-1}x + (t+1)^{-1}v \in U^0 \cap V$, which is impossible. Let p be the Minkowski functional of the set W^0. Then $p(v_0) \geqslant 1$. On the one-dimensional subspace of vectors of the form tv_0 we define a linear function $l_0(tv_0) = tp(v_0)$. By the Hahn–Banach theorem l_0 extends to a linear function l on X such that $l \leqslant p$. Then $l(w) \leqslant 1$ if $w \in W^0$, whence $l(w) \leqslant 1$ for all $w \in W$, because $\lambda w \in W^0$ if $\lambda \in (0,1)$. Since $l(v_0) = l_0(v_0) = p(v_0) \geqslant 1$, the functional l separates the set W and v_0. Hence l separates $U - V$ and $\{0\}$, but then l separates the sets U and V. □

Note that in the previous theorem no topology was used: the algebraic kernel is defined in purely algebraic terms. Applying to normed spaces, we obtain the following assertion (see also Example 8.3.11).

6.3.8. Corollary. *Suppose that two convex sets U and V in a real normed space X are disjoint and U is open. Then there exists a nonzero continuous linear function separating U and V.*

PROOF. We observe that the algebraic kernel of U coincides with U by the assumption that U is open. Hence there exists a nonzero linear function l separating U and V. This function is automatically continuous in case of an open set U. This follows from an obvious observation: if a linear function l is bounded from above or below on a nonempty open set, then it is continuous. Indeed, the function l is bounded from above or below on some ball of radius $r > 0$. Hence it is bounded from above or below on the ball of radius r centered at the origin, which gives the boundedness in absolute value on this ball. □

6.3.9. Corollary. *If V is a closed convex set in a real normed space X and $x \notin V$, then there exists $l \in X^*$ with $l(x) > \sup_{v\in V} l(v)$. If V is a linear subspace, then one can take l such that $l(x) = 1$ and $l|_V = 0$.*

PROOF. We can assume that $x = 0$. There is an open ball U centered at the origin such that $U \cap V = \varnothing$. By the previous theorem there exists a nonzero functional $l \in X^*$ with $\inf_{u\in U} l(u) \geqslant \sup_{v\in V} l(v)$. Then $\inf_{u\in U} l(u) < l(0) = 0$, since otherwise $l = 0$. If V is linear, then $l|_V = 0$ (otherwise the supremum is infinite). □

6.4. Applications of the Hahn–Banach Theorem

Some interesting applications of the Hahn–Banach theorem have already been discussed above: the separation theorem. Another important application is the proof of the fact (not a priori obvious) that the topological dual to the infinite-dimensional normed space is nonzero.

6.4.1. Theorem. *For every nonzero element x of a normed space X there exists a functional l such that $\|l\| = 1$ and $l(x) = \|x\|$.*

PROOF. On the one-dimensional space generated by x we set $l_0(tx) = t\|x\|$. Then $l_0(x) = \|x\|$ and $\|l_0\| = 1$. It remains to extend l_0 to X with the preservation of its norm. □

With the aid of a similar reasoning it is easy to establish that in the case of an infinite-dimensional space X for every n there exist vectors $x_1, \ldots, x_n \in X$ and functionals $l_1, \ldots, l_n \in X^*$ such that $l_i(x_j) = \delta_{ij}$. In particular, the dual space is also infinite-dimensional.

6.4.2. Corollary. *Let X_0 be a finite-dimensional subspace of a normed space X. Then X_0 is topologically complemented in X, i.e., there exists a closed linear subspace X_1 such that X is the direct algebraic sum of X_0 and X_1 and the natural algebraic projections P_0 and P_1 to X_0 and X_1 are continuous.*

PROOF. As noted above, one can find a basis $x_1, \ldots, x_n$ in the space X_0 and elements $l_i \in X^*$ with $l_i(x_j) = \delta_{ij}$. Set

$$X_1 := \bigcap_{i=1}^{n} \operatorname{Ker} l_i, \quad P_0 x := \sum_{i=1}^{n} l_i(x) x_i, \quad P_1 x := x - P_0 x.$$

For every j we have $P_0 x_j = l_j(x_j) x_j = x_j$. Note that $P_0|_{X_1} = 0$, $X_0 \cap X_1 = \{0\}$. In addition, we have $X = X_0 \oplus X_1$, because $x - P_0 x \in X_1$ by the equalities $l_j(x - P_0 x) = l_j(x) - l_j(x) l_j(x_j) = 0$. The continuity of P_0 and P_1 is obvious from their definitions. It is also clear that P_0 and P_1 coincide with the algebraic projections to X_0 and X_1. □

We now construct an isometric embedding of any normed space X into its second dual X^{**}. For every $x \in X$, we consider the functional $J_x \colon f \mapsto f(x)$ on the space X^*.

6.4.3. Proposition. *The mapping $J \colon x \mapsto J_x$ is a linear isometric embedding of X into X^{**}.*

PROOF. The linearity of J is obvious. Since

$$|J_x(f)| = |f(x)| \leqslant \|x\| \quad \text{whenever } \|f\| \leqslant 1,$$

we have $\|J_x\| \leqslant \|x\|$. On the other hand, as shown above, if $x \neq 0$, there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$, i.e., $J_x(f) = \|x\|$, whence we obtain that $\|J_x\| \geqslant \|x\|$. □

If $J(X) = X^{**}$, then the space X is called *reflexive*. Below we give examples of reflexive and nonreflexive spaces. One should bear in mind that the reflexivity of a space X is not equivalent to the existence of an isometry between X and X^{**} (Exercise 6.10.180 contains a counter-example: the famous James space); it is required that the canonical mapping J be an isometry onto all of X^{**}.

Combining this proposition with the Banach–Steinhaus theorem, we obtain an important assertion about the boundedness of weakly bounded sets.

6.4.4. Definition. *A set A in a normed space is called weakly bounded if*

$$\sup_{x\in A} |l(x)| < \infty$$

for every continuous linear functional l.

In the real case it suffices to have such an estimate without absolute value, since $-l$ is a continuous functional as well.

6.4.5. Theorem. *A set in a normed space is weakly bounded precisely when it is bounded in norm.*

PROOF. The weak boundedness of the set A means that the family of functionals J_x, where $x \in A$, is bounded on every element of X^*. Since X^* is Banach, this family is bounded in the norm of X^{**} by the Banach–Steinhaus theorem. According to the proposition above the set A is norm bounded. The converse assertion is obvious. □

Yet another useful corollary of the existence of an isometric embedding of X into X^{**} is the following result.

6.4.6. Proposition. *Every normed space X possesses a unique* (*up to a linear isometry*) *completion that is a Banach space.*

PROOF. For such a completion we can take the closure of the image of X under the embedding into X^{**}. It should be noted that using the completion constructed earlier in the category of general metric spaces, one can obtain a space that is not linear. The uniqueness of a completion up to a linear isometry is easily verified. □

In the case of a separable normed space, by using the Hahn–Banach theorem it is easy to obtain a countable set of functionals separating points.

6.4.7. Proposition. *Let X be a separable normed space. Then there exists a countable set of functionals $l_n \in X^*$ such that the equality $l_n(x) = 0$ for all n implies the equality $x = 0$.*

PROOF. Let $\{x_n\}$ be a countable everywhere dense set in X. Assuming that $X \neq 0$, for every n we find $l_n \in X^*$ with $l_n(x_n) = \|x_n\|$ and $\|l_n\| = 1$. Let $l_n(x) = 0$ for all n. Let us fix $\varepsilon > 0$ and find x_m with $\|x - x_m\| \leqslant \varepsilon$. Then $\|x_m\| = l_m(x_m) = l_m(x_m - x) \leqslant \|x_m - x\| \leqslant \varepsilon$, whence $\|x\| \leqslant 2\varepsilon$. Hence we have $x = 0$. □

The existence of functionals separating points will be used in the theorem on universality of the space $C(K)$ proved in §6.7.

By using the Hahn–Banach theorem we constructed functionals having maxima on the unit ball. On the other hand, we have encountered examples of functionals that do not attain maxima. In this connection we mention the following result due to Bishop and Phelps (its proof can be found, for example, in Diestel [**147**, Chapter 1]; a stronger assertion can be found in Bollobás [**77**, Chapter 8]).

6.4.8. Theorem. *Let C be a nonempty closed bounded convex set in a real Banach space X. Then the set of functionals in X^* attaining their maximum on C is everywhere dense in X^*.*

In particular, this assertion is true for closed balls. Not all functionals attain their maxima on balls: this is true only in reflexive spaces (see Theorem 6.10.10).

Note that for every normed space X with the completion $\overline{X}$, the duals X^* and $\overline{X}^*$ coincide in the sense that every functional from X^* extends uniquely by continuity to a functional from $\overline{X}^*$, moreover, every element of $\overline{X}^*$ is obtained in this way from its restriction to X. If Banach spaces X and Y are linearly homeomorphic by means of an operator J, then the mapping $l \mapsto l \circ J$ is a homeomorphism from Y^* onto X^*. However, one should bear in mind that there exist non-isomorphic (linearly topologically) Banach spaces X and Y with linearly homeomorphic duals. As an example one can take l^1 and $L^1[0,1]$. The absence of linear isomorphisms between them is the subject of Exercise 6.10.104 and the fact that their duals l^∞ and $L^\infty[0,1]$ are isomorphic is Pełczyński's theorem, a proof of which can be read in Albiac, Kalton [**9**, Theorem 4.3.10].

Let us give less obvious examples of positively homogeneous convex functions that are useful for constructing some interesting linear functions.

6.4.9. Example. The following functions p are positively homogeneous convex:

(i) let X be the space of all bounded real sequences $x = (x_n)$ and let

$$p(x) = \inf S(x, a_1, \ldots, a_n), \quad S(x, a_1, \ldots, a_n) := \sup_{k \geqslant 1} \frac{1}{n} \sum_{i=1}^n x_{k+a_i},$$

where inf is taken over all natural numbers n and all finite collections of numbers $a_1, \ldots, a_n \in \mathbb{N}$;

(ii) let X be the space of all bounded real functions on the real line and let

$$p(f) = \inf S(f, a_1, \ldots, a_n), \quad S(f, a_1, \ldots, a_n) := \sup_{t \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n f(t + a_i),$$

where inf is taken over all natural numbers n and all finite collections of numbers $a_1, \ldots, a_n \in \mathbb{R}$;

(iii) let X be the space of all bounded real sequences $x = (x_n)$ and let

$$p(x) = \inf S(x, a_1, \ldots, a_n), \quad S(x, a_1, \ldots, a_n) := \limsup_{k \to \infty} \frac{1}{n} \sum_{i=1}^n x_{k+a_i},$$

where inf is taken over all natural numbers n and all finite collections of numbers $a_1, \ldots, a_n \in \mathbb{N}$.

PROOF. Assertion (i) follows from (ii), so we prove the latter. It is clear that $|p(f)| < \infty$ and $p(\alpha f) = \alpha p(f)$ if $\alpha \geqslant 0$. Let $f, g \in X$ and $\varepsilon > 0$. Find $a_1, \ldots, a_n, b_1, \ldots, b_m$ such that

$$\sup_{t \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n f(t + a_i) < p(f) + \varepsilon, \quad \sup_{t \in \mathbb{R}} \frac{1}{m} \sum_{i=1}^m g(t + b_i) < p(g) + \varepsilon.$$

We observe that the quantity $\sup\limits_{t\in\mathbb{R}}(mn)^{-1}\sum_{j=1}^m\sum_{i=1}^n(f+g)(t+a_i+b_j)$ does not exceed the sum

$$\sup_{t\in\mathbb{R}}\frac{1}{m}\sum_{j=1}^m\frac{1}{n}\sum_{i=1}^n f(t+a_i+b_j)+\sup_{t\in\mathbb{R}}\frac{1}{n}\sum_{i=1}^n\frac{1}{m}\sum_{j=1}^m g(t+a_i+b_j).$$

We have $n^{-1}\sum_{i=1}^n f(t+a_i+b_j)\leqslant S(f,a_1,\ldots,a_n)$ for any fixed t and b_j, whence

$$\sup_{t\in\mathbb{R}}\frac{1}{m}\sum_{j=1}^m\frac{1}{n}\sum_{i=1}^n f(t+a_i+b_j)\leqslant S(f,a_1,\ldots,a_n).$$

From a similar estimate for g we obtain

$$p(f+g)\leqslant S(f,a_1,\ldots,a_n)+S(g,b_1,\ldots,b_m)<p(f)+p(g)+2\varepsilon,$$

which gives $p(f+g)\leqslant p(f)+p(g)$ because ε was arbitrary. The proof of assertion (iii) is completely similar. □

We now apply the established facts for constructing some curious set functions.

6.4.10. Example. On the σ-algebra of all subsets of $\mathbb{N}$ there exists a nonnegative additive function ν that equals zero on all finite sets and equals 1 on $\mathbb{N}$; in particular, the function ν is not countably additive.

PROOF. In the space X of bounded sequences with the function p from assertion (iii) in the previous example we take the subspace X_0 of sequences having a limit. Set $l(x)=\lim\limits_{n\to\infty}x_n$ if $x\in X_0$. Then $l(x)=p(x)$, since $\limsup\limits_{k\to\infty}n^{-1}\sum_{i=1}^n x_{k+a_i}=\lim\limits_{k\to\infty}x_k$ for all fixed n and $a_1,\ldots,a_n$. Let us extend l to a linear function $\widehat{l}$ on X with $\widehat{l}\leqslant p$. If $x\in X$ and $x_n\leqslant 0$ for all n, then $p(x)\leqslant 0$ and hence $\widehat{l}(x)\leqslant 0$. Hence $\widehat{l}(x)\geqslant 0$ if $x_n\geqslant 0$. If $x=(x_1,\ldots,x_n,0,0,\ldots)$, then $\widehat{l}(x)=l(x)=0$. Finally, $\widehat{l}(1,1,\ldots)=1$. For every $E\subset\mathbb{N}$ set $\nu(E):=\widehat{l}(I_E)$, where I_E is the indicator function of E, i.e., the sequence on the nth position of which one has 1 or 0 depending on whether n belongs to E or not. To finite sets there correspond finite sequences, so ν vanishes on them. On all of $\mathbb{N}$ the value of ν is 1, and the additivity of ν follows from the additivity of $\widehat{l}$ and the equality $I_{E_1\cup E_2}=I_{E_1}+I_{E_2}$ for disjoint E_1 and E_2. The failure of the countable additivity is obvious. □

6.4.11. Example. On the space X of all bounded real functions with bounded support on the real line there exists a linear function L that coincides with the Lebesgue integral on all Lebesgue integrable functions and has the following properties: $L(f)\geqslant 0$ if $f\geqslant 0$, $|L(f)|\leqslant\sup_t|f(t)|$, $L\bigl(f(\cdot+h)\bigr)=L(f)$ for all $f\in X$ and $h\in\mathbb{R}^1$, where $f(\cdot+h)(t)=f(t+h)$.

PROOF. First we construct such functional on the space X_1 of bounded functions with period 1. Let us consider on X_1 the function p from Example 6.4.9(ii). On the subspace X_0 of functions integrable over the period we define L as the Lebesgue integral and observe that $L(f) \leqslant p(f)$ if $f \in X_0$ by Exercise 3.12.26. Let us extend L to X by the Hahn–Banach theorem. We have $L(-f) = -L(f) \leqslant p(-f)$, whence

$$-p(-f) \leqslant L(f) \leqslant p(f) \quad \forall f \in X_1.$$

If $f \geqslant 0$, then $p(-f) \leqslant 0$ by the definition of p and hence $L(f) \geqslant 0$. It is clear that $|L(f)| \leqslant \sup_t |f(t)|$, since $p(f) \leqslant \sup_t |f(t)|$. We show that $L(f) = L\bigl(f(\cdot + h)\bigr)$ for all $f \in X_1$, $h \in \mathbb{R}^1$. Let us set $g(t) = f(t+h) - f(t)$ and verify that $L(g) = 0$. Let $a_k = (k-1)h$ if $k = 1, \ldots, n$. Then $\sum_{i=1}^{n} g(t + a_i) = f(t + nh) - f(t)$, whence

$$p(g) \leqslant S(g, a_1, \ldots, a_n) = \sup_t n^{-1}[f(t+nh) - f(t)] \to 0 \text{ as } n \to \infty.$$

Thus, $p(g) \leqslant 0$. Similarly $p(-g) \leqslant 0$, whence $L(g) = 0$. Now for a bounded function f with support in $[-n, n)$ we set $L(f) := \sum_{j=-n}^{n-1} L(f_j)$, where f_j is the 1-periodic extension of the restriction of the function f to $[j, j+1)$. It is not difficult to verify that we have obtained the desired functional. □

Both examples are due to Banach.

Letting $\zeta(A) := L(I_A)$ for a bounded set A, we obtain an additive nonnegative set function extending Lebesgue measure to bounded sets and invariant with respect to translations. By the way, if we take for X_0 the one-dimensional space of constants, then the same reasoning gives yet another linear functional on X_1 equal to 1 at 1, nonnegative on nonnegative functions and invariant with respect to shifts of the argument of functions. By a similar reasoning, on every set T on which there is an action of a commutative group of bijections $\mathcal{G}$, one can construct a nonnegative additive function ζ with $\zeta(T) = 1$ invariant with respect to $\mathcal{G}$. In this way Banach constructed an additive area on the class of all bounded sets that coincides with Lebesgue measure on measurable sets and is invariant with respect to isometries. However, for $\mathbb{R}^3$ such an extension does not exist, which was proved by Hausdorff. Hence the commutativity of the group $\mathcal{G}$ is important.

Here is one more example of application of the Hahn–Banach theorem.

6.4.12. Example. Let X, Y be normed spaces, let $D \subset X$ be a linear subspace, $T\colon D \to Y$ a linear mapping (not necessarily continuous), and let l be a linear function on D such that $|l(x)| \leqslant \|Tx\|_Y$ for all $x \in D$. Then there exists a functional $f \in Y^*$ for which $l(x) = f(Tx)$ for all $x \in D$.

PROOF. If $\operatorname{Ker} T = 0$, then $l_0(y) = l(T^{-1}y)$ is a linear functional on $T(D)$ and $|l_0(y)| \leqslant \|y\|_Y$. Let us extend l_0 to a functional $f \in Y^*$. Then $l(x) = f(Tx)$ for all $x \in D$. If $\operatorname{Ker} T \neq 0$, then we take a linear subspace D_1 in D such that $D = \operatorname{Ker} T \oplus D_1$. As we have proved, there exists $f \in Y^*$ for which $l(x) = f(Tx)$ for all $x \in D_1$. This is also true for $x \in \operatorname{Ker} T$, since $|l(x)| \leqslant \|Tx\|_Y$. Hence this is true for all $x \in D$. □

This result will be used below in Exercise 6.5.3 for representing linear functionals by means of inner products.

6.5. Duals to Concrete Spaces

Here we describe the dual spaces to the most important Banach spaces. We start with the Riesz theorem on a general form of a continuous linear functional on a Hilbert space.

6.5.1. Theorem. (THE RIESZ THEOREM) *Let H be a real or complex Hilbert space. Then for every $v \in H$ the formula*

$$l_v(x) = (x, v)$$

defines a continuous linear functional on H and $\|l_v\| = \|v\|$.
Conversely, every functional $l \in H^*$ is represented in this way, and the mapping $v \mapsto l_v$ is an isometry that is linear in the real case and conjugate-linear in the complex case.

PROOF. The equality $\|l_v\| = \|v\|$ is obvious from the Cauchy inequality $|(x, v)| \leqslant \|x\|\|v\|$ and the equality $l_v(v) = \|v\|^2$. Let $l \in H^*$. If $l = 0$, then $v = 0$. We shall assume that $H_0 := l^{-1}(0) \neq H$. Let us take a unit vector $e \perp H_0$ and set $v := \overline{l(e)}e$. Then for all $x \in H$ we have $l(x) = (x, v)$, since this true for all $x \in H_0$ and for $x = v$ by the equalities $l(v) = |l(e)|^2$, $(v, v) = |l(e)|^2$. The last assertion of the theorem is obvious. □

In the complex case the Riesz identification of H with H^* is conjugate-linear, but not linear. However, there is also a linear isometry between H and H^*. Indeed, by using an orthonormal basis $\{e_\alpha\}$ in H we can define the conjugation $v \mapsto \overline{v}$ by sending $\sum_\alpha c_\alpha e_\alpha$ to $\sum_\alpha \overline{c_\alpha} e_\alpha$. Then $v \mapsto l_{\overline{v}}$ is the desired linear isometry.

It follows from the Riesz theorem that every continuous linear functional F on H^* has the form $F(l) = l(a)$, where $a \in H$. Indeed, the function $v \mapsto \overline{F(l_v)}$ is linear and continuous. Hence $\overline{F(l_v)} = (v, a)$ for some $a \in H$, i.e., we have $F(l_v) = (a, v) = l_v(a)$. Therefore, H is reflexive.

6.5.2. Remark. For every real Euclidean space E (not necessarily complete) the linear mapping $J\colon E \to E^*$, $v \mapsto l_v$, where $l_v(x) = (x, v)$, preserves the distances. Since E^* is always complete, this mapping gives a completion of E in the category of Euclidean spaces. For this it suffices to observe that on the closure of $J(E)$ in E^* we have not only the norm, but also the inner product: if $f = \lim\limits_{n\to\infty} J(v_n)$ and $g = \lim\limits_{n\to\infty} J(w_n)$, then the limit $(f, g) := \lim\limits_{n\to\infty} (w_n, v_n)$ exists, does not depend on the choice of convergent sequences, is linear in each argument (since we have $J(\lambda v_n) = \lambda J(v_n)$ and $(\lambda w_n, v_n) = \lambda(w_n, v_n)$). We can also set $(f, g) := [\|f + g\|^2 - \|f\|^2 - \|g\|^2]/2$ and use v_n and w_n to make sure that we have obtained an inner product. Thus, the closure $\overline{J(E)}$ of $J(E)$ is a Hilbert space. Indeed, $\overline{J(E)} = E^*$, since after identification of E with $J(E)$ and $\overline{J(E)}^*$ with $\overline{J(E)}$ we obtain $E^* = \overline{J(E)}^* = \overline{J(E)}$. In a similar way one can construct completions of complex Euclidean spaces.

6.5.3. Example. Let D be a linear subspace in a Hilbert space H, $T\colon D \to H$ a linear mapping, l a linear function on D such that $|l(x)| \leqslant \|Tx\|$ for all $x \in D$. Then there exists $v \in H$ for which $l(x) = (Tx, v)$ for all $x \in D$. This follows from Example 6.4.12.

Let us apply the Riesz theorem to prove the Radon–Nikodym theorem.

6.5.4. Example. Let $\mu \geqslant 0$ and $\nu \geqslant 0$ be finite measures on a space $(X, \mathcal{A})$ and $\nu \ll \mu$. Let us consider the measure $\lambda = \mu + \nu$. Every function φ from $\mathcal{L}^1(\lambda)$ belongs to $\mathcal{L}^1(\mu)$ and its integral against the measure μ does not change if we redefine φ on a set of λ-measure zero. Therefore, the linear function

$$L(\varphi) = \int_X \varphi \, d\mu$$

is well-defined on the real space $L^2(\lambda)$ (does not depend on the choice of a representative of φ). By the Cauchy–Bunyakovskii inequality

$$|L(\varphi)| \leqslant \int_X |\varphi| \, d\lambda \leqslant \|1\|_{L^2(\lambda)} \|\varphi\|_{L^2(\lambda)}.$$

The Riesz theorem gives a function $\psi \in \mathcal{L}^2(\lambda)$ such that

$$\int_X \varphi \, d\mu = \int_X \varphi\psi \, d\lambda \tag{6.5.1}$$

for all $\varphi \in L^2(\lambda)$. We shall deal with its $\mathcal{A}$-measurable version. Substituting $\varphi = I_A$, where $A \in \mathcal{A}$, we find that $\mu = \psi \cdot \lambda$, $\nu = (1 - \psi) \cdot \lambda$. Let us prove that the function $(1 - \psi)/\psi$ can be taken for $d\nu/d\mu$. Let $\Omega = \{x\colon\ \psi(x) \leqslant 0\}$, $\Omega_1 = \{x\colon\ \psi(x) > 1\}$. Then $\Omega, \Omega_1 \in \mathcal{A}$. Substituting in (6.5.1) the functions $\varphi = I_\Omega$ and $\varphi = I_{\Omega_1}$, we obtain

$$\mu(\Omega) = \int_\Omega \psi \, d\lambda \leqslant 0, \quad \mu(\Omega_1) = \int_{\Omega_1} \psi \, d\lambda > \lambda(\Omega_1)$$

if $\mu(\Omega_1) > 0$, whence $\mu(\Omega) = 0$ and $\mu(\Omega_1) = 0$, since $\mu(\Omega_1) \leqslant \lambda(\Omega_1)$. Then the function f defined by the equality

$$f(x) = \frac{1 - \psi(x)}{\psi(x)} \quad \text{if } x \notin \Omega \cup \Omega_1, \quad f(x) = 0 \quad \text{if } x \in \Omega \cup \Omega_1$$

is nonnegative and $\mathcal{A}$-measurable. The function f is integrable with respect to μ. Indeed, the functions $f_n = f I_{\{\psi \geqslant 1/n\}}$ are bounded and increase pointwise to f, moreover,

$$\int_X f_n \, d\mu = \int_X I_{\{\psi \geqslant 1/n\}} (1 - \psi) \, d\lambda = \int_X I_{\{\psi \geqslant 1/n\}} \, d\nu \leqslant \nu(X).$$

The Beppo Levi theorem yields convergence of $\{f_n\}$ to f in $L^1(\mu)$. Finally, for every $A \in \mathcal{A}$ we have $I_A I_{\{\psi \geqslant 1/n\}} \to I_A$ μ-a.e., hence also ν-a.e. (only here we use the absolute continuity of ν with respect to μ). By convergence of $\{f_n\}$ to f in $L^1(\mu)$ and the equality $I_{\{\psi \geqslant 1/n\}} \cdot \nu = I_{\{\psi \geqslant 1/n\}} (1 - \psi) \cdot \lambda = I_{\{\psi \geqslant 1/n\}} f \cdot \mu$ we find that

$$\nu(A) = \lim_{n \to \infty} \int_X I_A I_{\{\psi \geqslant 1/n\}} \, d\nu = \lim_{n \to \infty} \int_X I_A I_{\{\psi \geqslant 1/n\}} f \, d\mu = \int_A f \, d\mu.$$

Let us turn to another theorem due to F. Riesz about a general form of a continuous linear functional on the space $C[a, b]$.

6.5.5. Theorem. (THE RIESZ THEOREM FOR C) *The general form of a continuous linear function on the real or complex space $C[a, b]$ with its sup-norm* *is given by the following integral formula*:

$$l(x) = \int_{[a,b]} x(t)\,\mu(dt),$$

where μ *is a bounded Borel measure on* $[a,b]$ (*complex in the complex case*), *moreover,* $\|l\| = \|\mu\|$.

PROOF. For simplification of formulas we consider the real case. By the Hahn–Banach theorem every continuous linear functional l on $C[a,b]$ extends with the same norm to the space $B[a,b]$ of all bounded functions on $[a,b]$. The extension is also denoted by l. Set

$$F(s) := l(I_{[a,s)}) \text{ if } s \in (a,b],\ F(s) = 0 \text{ if } s \leqslant a,\ F(s) = l(I_{[a,b]}) \text{ if } s > b.$$

We show that the function F is of bounded variation. Let $a = t_0 < \cdots < t_n$, where $t_{n-1} \leqslant b$ and $t_n \geqslant b$. Set $\varepsilon_i := \operatorname{sign}\bigl(F(t_i) - F(t_{i-1})\bigr)$ and $[t_{n-1}, t_n) = [t_{n-1}, b]$ if $t_n > b$. Then

$$\sum_{i=1}^n |F(t_i) - F(t_{i-1})| = \sum_{i=1}^n \varepsilon_i \bigl(F(t_i) - F(t_{i-1})\bigr)$$
$$= l\Bigl(\sum_{i=1}^n \varepsilon_i I_{[t_{i-1},t_i)}\Bigr) \leqslant \|l\| \Bigl\|\sum_{i=1}^n \varepsilon_i I_{[t_{i-1},t_i)}\Bigr\| \leqslant \|l\|,$$

since the function $\sum_{i=1}^n \varepsilon_i I_{[t_{i-1},t_i)}$ can assume only the values $-1, 1, 0$. Thus, we have $V(F, \mathbb{R}) \leqslant \|l\|$. The function F has at most a countable set T of discontinuity points (see §4.2), and these points are jumps. We redefine F at these points in order to obtain a left continuous function F_0. It is readily seen that $V(F_0, \mathbb{R}) \leqslant V(F, \mathbb{R})$.

Let now x be a function continuous on $[a,b]$. We fix $\varepsilon > 0$ and take $\delta > 0$ with the following properties: $|x(t) - x(s)| \leqslant \varepsilon$ whenever $|t - s| \leqslant \delta$ and

$$\Bigl|\int_a^b x(t)\,dF_0(t) - \sum_{i=1}^n x(t_i)\bigl(F_0(t_i) - F_0(t_{i-1})\bigr)\Bigr| \leqslant \varepsilon$$

if points $a = t_0 < t_1 < \cdots < t_n$ are such that $|t_i - t_{i-1}| \leqslant \delta$, $t_{n-1} \leqslant b$, $t_n > b$. We take such points outside T (at these points F_0 and F are equal) and take a step function ψ equal to $x(t_{i-1})$ on $[t_{i-1}, t_i)$, $i = 1, \ldots, n$. Clearly, we have $\|x - \psi\| \leqslant \varepsilon$. Hence we obtain $|l(x) - l(\psi)| \leqslant \varepsilon\|l\|$. Finally, we have $l(\psi) = \sum_{i=1}^n x(t_{i-1})\bigl(F_0(t_i) - F_0(t_{i-1})\bigr)$, whence

$$\Bigl|\int_a^b x(t)\,dF_0(t) - l(x)\Bigr| \leqslant \varepsilon(1 + \|l\|).$$

Since ε was arbitrary, $l(x)$ is the Stieltjes integral of x against F_0. This integral equals the Lebesgue integral with respect to the Borel measure μ generated by F_0 (see §4.2). It is readily seen that the Stieltjes integral of x against F_0 is not greater than $V(F_0, \mathbb{R}) \sup_t |x(t)|$, whence $\|l\| \leqslant V(F_0, \mathbb{R})$, so $\|l\| = V(F_0, \mathbb{R})$. The equality $\|\mu\| = V(F_0, \mathbb{R})$ is proved in Proposition 4.2.9. □

Actually, as shown in §6.10(vii), the Riesz theorem remains in force for every compact space, so it is possible to use more advanced techniques as compared to the crude construction presented above. The reader may ask the following question: is not it possible to obtain the required measure μ at once as the action of the extension of l on the indicator functions? It turns out that the answer is negative: the obtained set function can fail to be countably additive (see Exercise 6.10.100).

6.5.6. Theorem. (i) *The general form of a continuous linear function on* c_0 *is given by the series*

$$l(x) = \sum_{n=1}^{\infty} y_n x_n, \quad \textit{where } y = (y_n) \in l^1 \textit{ and } \|l\| = \|y\|_{l^1}.$$

(ii) *Let* $1 < p < \infty$. *The general form of a continuous linear function on* l^p *is given by the series*

$$l(x) = \sum_{n=1}^{\infty} y_n x_n, \quad \textit{where } y = (y_n) \in l^q,\ 1/p + 1/q = 1 \textit{ and } \|l\| = \|y\|_{l^q}.$$

(iii) *The general form of a continuous linear function on* l^1 *is given by the series*

$$l(x) = \sum_{n=1}^{\infty} y_n x_n, \quad \textit{where } y = (y_n) \in l^\infty \textit{ and } \|l\| = \|y\|_{l^\infty}.$$

PROOF. It is clear that every element $y \in l^1$ defines a functional l on c_0 by this formula and $\|l\| \leqslant \|y\|_{l^1}$. Since for the vector

$$x = (\operatorname{sgn} y_1, \ldots, \operatorname{sgn} y_n, 0, 0, \ldots)$$

we have $\|x\| = 1$ and $l(x) = \sum_{i=1}^{n} |y_i|$, we actually have $\|l\| = \|y\|_{l^1}$.

Conversely, let l be a continuous linear functional on c_0. Set $y_n := l(e_n)$, where e_n is the sequence with 1 at the nth position and 0 at all other positions. Then $\sum_{i=1}^{n} |y_i| = l(x) \leqslant \|l\|$ for $x = (\operatorname{sgn} y_1, \ldots, \operatorname{sgn} y_n, 0, 0, \ldots)$, whence $y \in l^1$. The vector y defines the functional l_0 that coincides with l on finite linear combinations of the vectors e_n. Since such combinations are dense in c_0 and both functionals are continuous, we obtain the equality $l = l_0$. This proves assertion (i). Assertions (ii) and (iii) are proved completely similarly, one only has to use the inequality $\sum_{n=1}^{\infty} |x_n y_n| \leqslant \|x\|_{l^p} \|y\|_{l^q}$. □

Thus, $c_0^{**} = l^\infty$. This equality gives one of a rather few known examples in which the second dual space is explicitly calculated for a non-reflexive space. Of course, forming direct sums of c_0 with reflexive spaces one can increase the number of examples, but without using c_0 one can hardly proceed. Note that the dual $(l^\infty)^*$ is not isomorphic to l^1, since is not separable (see Exercise 6.10.86).

We now consider the space L^p.

6.5.7. Theorem. *Let* μ *be a nonnegative* σ*-finite measure on a measurable space* $(\Omega, \mathcal{A})$.

(i) *Let* $1 < p < \infty$. *The general form of a continuous linear function on the real or complex space* $L^p(\mu)$ *is given by the formula*

$$l(x) = \int_\Omega x(t) y(t)\, \mu(dt), \quad \textit{where } y \in L^q(\mu),\ 1/p + 1/q = 1.$$

Moreover, $\|l\| = \|y\|_{L^q(\mu)}$.

(ii) *The general form of a continuous linear function on the real or complex space* $L^1(\mu)$ *is given by the formula*

$$l(x) = \int_\Omega x(t)y(t)\,\mu(dt), \quad \textit{where } y \in \mathcal{L}^\infty(\mu).$$

Moreover, $\|l\| = \|y\|_{L^\infty(\mu)}$.

PROOF. To simplify calculations we consider the real case. Hölder's inequality shows that every function $y \in L^q(\mu)$ defines on $L^p(\mu)$ a linear functional the norm of which does not exceed $\|y\|_{L^q(\mu)}$. If $p > 1$, we take for x the function $x(\omega) = \operatorname{sgn} y(\omega)|y(\omega)|^{q/p}$ and on account of the equality $q/p = q-1$ obtain

$$\|x\|^p_{L^p(\mu)} = \|y\|^q_{L^q(\mu)} \quad \text{and} \quad \int_\Omega x(\omega)y(\omega)\,\mu(d\omega) = \|y\|^q_{L^q(\mu)},$$

which shows that $\|l\| = \|y\|_{L^q(\mu)}$. In the case $p = 1$ we set $c := \|y\|_{L^\infty(\mu)}$. If $c = 0$, then $l = 0$. Let $c > 0$. The sets $E_n := \{\omega\colon\ c - 1/n \leqslant |y(\omega)| \leqslant c\}$ are of positive measure. Take the functions $x_n := I_{E_n} \operatorname{sgn} y/\mu(E_n)$ with norm 1. Then

$$l(x_n) = \int_\Omega x_n(\omega)y(\omega)\,\mu(d\omega) \geqslant c - 1/n$$

and hence $\|l\| \geqslant c - 1/n$, i.e., $\|l\| \geqslant c$. The opposite inequality is obvious.

Let l be a continuous linear functional on $L^p(\mu)$. Partitioning the space Ω into parts of finite measure, it is easy to reduce the general case to the case of a finite measure. Hence we further assume that $\mu(\Omega) < \infty$. The function

$$\nu(A) := l(I_A), \quad A \in \mathcal{A},$$

is a countably additive measure on $\mathcal{A}$. If $\mu(A) = 0$ for some set $A \in \mathcal{A}$, then $\nu(A) = 0$, i.e., the measure ν is absolutely continuous with respect to the measure μ. By the Radon–Nikodym theorem there exists a μ-integrable function y such that

$$l(I_A) = \nu(A) = \int_\Omega I_A(\omega)y(\omega)\,\mu(d\omega), \quad A \in \mathcal{A}.$$

Hence for every simple function x the quantity $l(x)$ equals the integral of xy. Therefore,

$$\left|\int_\Omega x(\omega)y(\omega)\,\mu(d\omega)\right| \leqslant \|l\|\|x\|_{L^p(\mu)}. \tag{6.5.2}$$

By a limiting procedure we extend this estimate to all bounded μ-measurable functions x. Let $p > 1$. We take for x the function $x(\omega) = \operatorname{sgn} y(\omega)|y(\omega)|^{q/p} I_{\{|y|\leqslant n\}}$, where $n \in \mathbb{N}$. This gives the estimate

$$\int_\Omega |y(\omega)|^q I_{\{|y|\leqslant n\}}(\omega)\,\mu(d\omega) \leqslant \|l\|\|yI_{\{|y|\leqslant n\}}\|^{q/p}_{L^q(\mu)}.$$

Thus, $\|yI_{\{|y|\leqslant n\}}\|^q_{L^q(\mu)} \leqslant \|l\|\|yI_{\{|y|\leqslant n\}}\|^{q/p}_{L^q(\mu)}$, hence by the equality $q = 1 + q/p$ we obtain the estimate $\|yI_{\{|y|\leqslant n\}}\|_{L^q(\mu)} \leqslant \|l\|$. By Fatou's theorem $\|y\|_{L^q(\mu)} \leqslant \|l\|$. The function y defines on $L^p(\mu)$ a continuous linear functional l_0 which coincides with l on all simple functions. By the continuity of both functionals and density of the set of simple functions in $L^p(\mu)$ with $p < \infty$ we obtain the equality $l = l_0$. Finally, in the case $p = 1$ from (6.5.2) we obtain the inequality

$$\int_A |y(\omega)|\,\mu(d\omega) \leqslant \|l\|\mu(A), \quad A \in \mathcal{A}.$$

Then $\mu\bigl(\omega\colon\ |y(\omega)| > \|l\|\bigr) = 0$, hence $\|y\|_{L^\infty(\mu)} \leqslant \|l\|$. □

Again, the dual $L^\infty[0,1]^*$ is not isomorphic to $L^1[0,1]$, since is not separable (see Exercise 6.10.86).

Note that there are cases when it is not convenient to identify the dual to a Hilbert space with the space itself, in spite of the existence of the natural isomorphism. For example, this is the case when one deals with duals to Sobolev spaces (see Chapter 9). Here is yet another typical example. Let Ω be the open unit disc in the complex plane and let $A^2(\Omega)$ be the Bergman space of functions holomorphic in Ω and belonging to $L^2(\Omega)$ (see Example 5.2.2). For every $\lambda \in \Omega$, the linear functional $\varphi \mapsto \varphi(\lambda)$ is continuous on $A^2(\Omega)$ (see estimate (5.2.1) in the aforementioned example). This functional is not written in the form (φ, ψ), although, of course, in accordance with the general theorem it can be represented in this form with some element $\psi \in A^2(\Omega)$. A similar situation arises for more general functionals of the form

$$\varphi \mapsto \int_K \varphi(z)\,\mu(dz),$$

where μ is a measure with compact support K in Ω.

6.6. The Weak and Weak-$*$ Topologies

Let E be a linear space and let F be some linear space of linear functions on E separating points in E in the following sense: for every $x \neq 0$, there exists $f \in F$ with $f(x) \neq 0$. In other words, for every pair of different points $x, y \in X$, there exists an element $f \in F$ with $f(x) \neq f(y)$. In this situation, when there are no norms or topologies, in many applications it is useful to introduce convergence on E in the following way: $x_n \to x$ if $f(x_n) \to f(x)$ for every $f \in F$.

Similarly, on F it is useful to consider the pointwise convergence, i.e., convergence $f_n(x) \to f(x)$ for every x. We now introduce a natural topology in which convergence of sequences has the indicated form. We warn the reader at once that a topology with such a property is not unique. However, the topology $\sigma(E, F)$ introduced below is natural in many respects. The objects defined and studied in this section belong actually to the theory of locally convex spaces, fundamentals of which are discussed in Chapter 8. However, the special features of weak topologies are so important that the violation of the deductive order of presentation undertaken here seems fully justified, moreover, it is even useful for the subsequent acquaintance with more general concepts.

We first define the following basis of neighborhoods of zero:

$$U_{f_1,\ldots,f_n,\varepsilon} := \bigl\{x \in E\colon\ |f_1(x)| < \varepsilon, \ldots, |f_n(x)| < \varepsilon\bigr\},$$

where $n \in \mathbb{N}$, $f_i \in F$, $\varepsilon > 0$. Next we introduce a basis of neighborhoods of an arbitrary point $a \in E$:

$$\begin{aligned} U_{f_1,\ldots,f_n,\varepsilon}(a) &:= U_{f_1,\ldots,f_n,\varepsilon} + a \\ &= \bigl\{x \in E\colon\ |f_1(x-a)| < \varepsilon, \ldots, |f_n(x-a)| < \varepsilon\bigr\}. \end{aligned}$$

Finally, we declare to be open the empty set and all possible unions of neighborhoods $U_{f_1,\ldots,f_n,\varepsilon}(a)$, i.e., in these neighborhoods one can vary points a as well as functionals f_i along with numbers n and ε. This topology is the restriction of the familiar topology of the pointwise convergence on $\mathbb{R}^F$ if points of E are regarded as functions on F.

6.6.1. Proposition. *The obtained class of sets is a Hausdorff topology, denoted by $\sigma(E,F)$ and called the weak topology generated by F. A sequence $\{x_n\}$ converges in this topology to x precisely when $f(x_n) \to f(x)$ for every $f \in F$.*

PROOF. We could refer to Example 1.6.5, but we repeat the reasoning. The indicated class contains E and the empty set and admits arbitrary unions. Let us verify that it admits finite intersections. For this it suffices to show that the intersection $V = U_{f_1,\ldots,f_n,\varepsilon}(a) \cap U_{g_1,\ldots,g_m,\delta}(b)$ belongs to $\sigma(E,F)$, i.e., every point $v \in V$ belongs to V along with a neighborhood of the form $U_{f_1,\ldots,f_n,g_1,\ldots,g_m,r}(v)$. To this end we pick $r > 0$ as follows:

$$r = \frac{1}{2}\min_{i,j}\bigl\{\varepsilon - |f_i(v-a)|, \delta - |g_j(v-b)|\bigr\}.$$

Then we obtain $U_{f_1,\ldots,f_n,g_1,\ldots,g_m,r}(v) \subset V$. Indeed, let $x \in U_{f_1,\ldots,f_n,g_1,\ldots,g_m,r}(v)$. We have

$$|f_i(x-a)| = |f_i(x-v) + f_i(v-a)| < r + |f_i(v-a)| < \varepsilon,$$

i.e., $x \in U_{f_1,\ldots,f_n,\varepsilon}(a)$. Similarly, $x \in U_{g_1,\ldots,g_m,\delta}(b)$. The Hausdorff property of this topology follows from the fact that F separates points of E: two points $a \neq b$ can be separated by a simple neighborhood of the form $U_{f,\varepsilon}(a)$ and $U_{f,\varepsilon}(b)$.

Suppose that a sequence $\{x_n\}$ converges in the topology $\sigma(E,F)$ to x. For every $f \in F$ and every $\varepsilon > 0$, there exists a number N such that $x_n \in U_{f,\varepsilon}(x)$ whenever $n \geqslant N$. This means that $f(x_n) \to f(x)$. Conversely, if, for every functional $f \in F$, this is fulfilled, then every neighborhood $U_{f_1,\ldots,f_n,\varepsilon}(x)$ contains all elements x_n starting from some number. □

A similar assertion is true for nets.

The topology $\sigma(E,F)$ possesses the following remarkable property.

6.6.2. Theorem. *The set of all linear functions on E continuous in the topology $\sigma(E,F)$ coincides with F, i.e., one has the equality $\bigl(E,\sigma(E,F)\bigr)^* = F$.*

PROOF. All functions from F are continuous on E by our construction of the topology. Let us show that every linear function l on E continuous in the topology $\sigma(E,F)$ is an element of F. By our assumption the set $\{x\colon\ |l(x)| < 1\}$ is open and contains the origin. Hence it contains some neighborhood of zero $U_{f_1,\ldots,f_n,\varepsilon}$. This means that l vanishes on the intersection of the kernels of the functionals f_i. We show that l is a linear combination of f_i. We use induction on n. Let $l = 0$ on the set $L_1 = f_1^{-1}(0)$. If $L_1 = E$, then $l = f_1 = 0$. If there is a vector $v \notin L_1$, then $l = l(v)f_1(v)^{-1}f_1$ by the linearity of the functionals l and f_1 and the fact that E is the sum of L_1 and the linear span of v. Suppose that our assertion is true for some $n-1 \geqslant 1$. Let us consider l on the kernel L_n of the functional f_n. By the inductive assumption there exist numbers $c_1, \ldots, c_{n-1}$ such that $l(x) = c_1 f_1(x) + \cdots + c_{n-1} f_{n-1}(x)$ for all $x \in L_n$. If $L_n = E$, then everything is proved. If there exists $v \notin L_n$, then $l = c_1 f_1 + \cdots + c_n f_n$, where we set $c_n := f_n(v)^{-1}\bigl(l(v) - c_1 f_1(v) - \cdots - c_{n-1} f_{n-1}(v)\bigr)$. □

Similarly we define the topology $\sigma(F, E)$ on F. To this end the elements of E must be considered as linear functionals on F, i.e., every $x \in E$ generates the functional $f \mapsto f(x)$. According to what we have proved above, convergence of sequences in the topology $\sigma(F, E)$ is the pointwise convergence. The space of continuous linear functions on F with the topology $\sigma(F, E)$ coincides with the original space E in the following sense: every linear function l continuous in the topology $\sigma(F, E)$ has the form $f \mapsto f(x)$ for some $x \in E$.

Let us consider two important examples.

6.6.3. Example. Let X be a normed space.

(i) Letting $E = X$ and $F = X^*$, we obtain the *weak topology* $\sigma(X, X^*)$ on X and the *weak convergence*. The dual to X with the topology $\sigma(X, X^*)$ remains X^*. However, in an infinite-dimensional space X the weak topology is *always strictly weaker* than the norm topology. This is seen from the fact that every basis neighborhood of zero $\{x \in X \colon |f_1(x)| < \varepsilon, \ldots, |f_n(x)| < \varepsilon\}$, $f_i \in X^*$, contains a linear subspace $\bigcap_{i=1}^n f_i^{-1}(0)$, which is infinite-dimensional in the case of any infinite-dimensional X. In particular, an open ball in X cannot contain weakly open sets. It is interesting that in spite of the difference of the weak topology and the norm topology, they can possess equal supplies of convergent sequences. This is the case for $X = l^1$ (Exercise 6.10.104).

(ii) Letting $E = X^*$ and taking X for F, we obtain the so-called *weak-$*$ topology* $\sigma(X^*, X)$ on X^* and the *weak-$*$ convergence*. The set of linear functions continuous in the topology $\sigma(X^*, X)$ is naturally identified with X, i.e., this set in the general case is smaller than the space X^{**}, as, for example, is the case for the space $X = c_0$, where we have $X^{**} = l^\infty$.

From the results of §6.4 we obtain the following assertion.

6.6.4. Proposition. *Every weakly convergent sequence in a normed space X is norm bounded.*

If X is complete, any sequence in X^* that converges in the weak-$*$ topology is norm bounded.

Of particular importance for applications is weak convergence in Hilbert spaces. Since here the dual can be identified with the space itself, the weak topology can be identified with the weak-$*$ topology.

6.6.5. Example. Let H be a separable Hilbert space with an orthonormal basis $\{e_n\}$. A sequence of vectors $h_k \in H$ converges weakly to a vector h

precisely when it is norm bounded and for every n the sequence of numbers (h_k, e_n) converges to (h, e_n).

PROOF. Let $\sup_n \|h_n\| < \infty$ and $(h_k, e_n) \to (h, e_n)$ for every n. It is clear that $(h_k, x) \to (h, x)$ for every x that is a finite linear combination of vectors e_n. This gives convergence on every element $x \in H$, since for every $\varepsilon > 0$ there exists a finite linear combination z of basis vectors with $\|x - z\| \leqslant \varepsilon$, whence we have $|(h_k, x) - (h_k, z)| \leqslant \varepsilon \sup_n \|h_n\|$. □

6.6.6. Example. A sequence of functions $f_n \in C[a, b]$ converges weakly to a function $f \in C[a, b]$ in and only if $\sup_n \|f_n\| < \infty$ and $f_n(t) \to f(t)$ for every point $t \in [a, b]$. The sufficiency of this condition is obvious from the Lebesgue dominated convergence theorem and the fact that $C[a, b]^*$ is the space of bounded Borel measures on $[a, b]$. The necessity is obvious from consideration of functionals $\varphi \mapsto \varphi(t)$ and norm boundedness of weakly convergent sequences.

The weak topology in the infinite-dimensional case is not metrizable. This has many different appearances.

6.6.7. Example. Let us consider the vectors $u_n = \ln(n + 1)\, e_n$ in l^2, where $\{e_n\}$ is the standard basis in l^2. The point 0 belongs to the closure of $\{u_n\}$ in the weak topology, but no subsequence in $\{u_n\}$ can converge weakly, since it is not norm bounded. In order to see that every weak neighborhood of zero contains points from $\{u_n\}$, we observe that for every fixed finite collection of vectors $v_1 = (v_{1,1}, \ldots, v_{1,j}, \ldots)$, $\ldots$, $v_m = (v_{m,1}, \ldots, v_{m,j}, \ldots)$ in l^2 and every $\varepsilon > 0$, there exists a number n satisfying the condition $\sum_{i=1}^{m} |v_{i,n}|^2 < \varepsilon |\ln(n + 1)|^{-2}$, which is obvious from divergence of the series of $|\ln(n + 1)|^{-2}$. Then we obtain the bound $|(v_1, u_n)| < \varepsilon, \ldots, |(v_m, u_n)| < \varepsilon$.

Let us note the following fact that is easily verified, but is unexpected at the first glance.

6.6.8. Theorem. *Let A be a linear mapping between normed spaces X and Y. The following conditions are equivalent*:

(i) *the mapping A is continuous*;

(ii) *the mapping A is continuous with respect to the weak topologies, i.e., as a mapping $A\colon \bigl(X, \sigma(X, X^*)\bigr) \to \bigl(Y, \sigma(Y, Y^*)\bigr)$*;

(iii) *if $x_n \to 0$ weakly, then $\{Ax_n\}$ is weakly convergent.*

PROOF. Let $\|A\| < \infty$. For a weak neighborhood of zero in Y of the form $V = \{y \in Y\colon |g_i(y)| < \varepsilon, g_1, \ldots, g_n \in Y^*\}$ we take the following weak neighborhood of zero: $U = \{x \in X\colon |f_i(x)| < \varepsilon, i = 1, \ldots, n\}$ in X, where $f_i := g_i \circ A \in X^*$. This gives the inclusion $A(U) \subset V$ and shows the weak continuity at zero, which obviously yields the continuity at all other points. It is clear that (ii) implies (iii). Suppose now that (iii) is fulfilled. Then $\|A\| < \infty$ by Corollary 6.1.3, since if $\|x_n\| \to 0$, then $x_n \to 0$ weakly, hence $\{Ax_n\}$ converges weakly by condition (iii), whence $\sup_n \|Ax_n\| < \infty$. □

It has already been noted that the weak topology is not metrizable if X is infinite-dimensional. Hence for nonlinear mappings (even with values in the real line) the weak sequential continuity can be strictly weaker than the continuity in the weak topology (see Exercise 6.10.167). However, on balls in l^2 the weak topology is metrizable (see §6.10(ii)).

We warn the reader that the established equivalence of continuity in the norm topology and the weak topology does not extend to intermediate topologies (Exercise 8.6.47).

Yet another remarkable property of the weak topology of a normed space is the fact that the supply of convex weakly closed sets is the same as in the original topology, although the supply of norm closed sets is larger than that of weakly closed sets (in the infinite-dimensional case).

6.6.9. Theorem. *A convex set V in a normed space X is closed in the weak topology precisely when it is norm closed. In addition, V is the intersection of all closed halfspaces of the form $\{x\colon\ l(x) \leqslant c\}$, $l \in X^*$ containing V.*

PROOF. Let V be norm closed and $u \notin V$. By Corollary 6.3.9 there exists $l \in X^*$ with $l(u) > c := \sup_{v\in V} l(v)$, i.e., $V \subset \Pi := \{x\colon\ l(x) \leqslant c\}$ and $u \notin \Pi$, which proves the last assertion that implies the first one at once. □

A convex set open in the norm topology can fail to have inner points in the weak topology (as, for example, an open ball). One should bear in mind that the established theorem does not extend to the weak-$*$ topology in the dual space (see Exercise 6.10.161).

6.6.10. Corollary. *Suppose that a sequence of vectors x_n in a normed space X converges weakly to $x \in X$. Then there exists a sequence of vectors v_n in the convex envelope of $\{x_n\}$ converging to x in norm.*

PROOF. The point x belongs to the closure V of the convex envelope of the sequence $\{x_n\}$ in the weak topology. It remains to observe that V is convex. Indeed, let $u, v \in V$, $t \in [0,1]$. For every basis neighborhood of zero U in the weak topology there exist points $u_1, v_1 \in \operatorname{conv}\{x_n\}$ with $u - u_1, v - v_1 \in U$. Then $tu_1 + (1-t)v_1 \in \operatorname{conv}\{x_n\}$ and $tu + (1-t)v - [tu_1 + (1-t)v_1] \in U$. Since U was arbitrary, we obtain $tu + (1-t)v \in V$, which proves the weak closedness of the set V. □

By analogy with the topology of the pointwise convergence on the space of functionals one can introduce a topology on the space of operators. Let X and Y be normed spaces. The space of operators $\mathcal{L}(X, Y)$ has, of course, the weak topology of a normed space. However, the *weak operator topology* is an even weaker topology in which the basis of neighborhoods of zero has the form

$$\{A \in \mathcal{L}(X, Y)\colon\ |l_i(Ax_i)| < \varepsilon,\ i = 1, \ldots, n\}, \quad x_i \in X,\ l_i \in Y^*.$$

If $X = Y$ are Hilbert spaces, then such neighborhoods are determined by “matrix elements” (Au_i, v_i). In the infinite-dimensional case, the weak operator topology is weaker than the topology $\sigma\bigl(\mathcal{L}(X), \mathcal{L}(X)^*\bigr)$, because not every continuous functional on $\mathcal{L}(X)$ is a finite linear combination of matrix elements. One more useful topology on $\mathcal{L}(X, Y)$ corresponds to the pointwise convergence of operators. It is called the *strong operator topology*. The corresponding neighborhoods of zero have the form

$$\{A \in \mathcal{L}(X,Y)\colon \|Ax_i\| < \varepsilon,\, i = 1,\ldots,n\}, \quad x_i \in X.$$

The strong and weak operator topologies are also considered in Exercises 6.10.187 and 7.10.118.

6.7. Compactness in the Weak-$*$ Topology

This section contains two important results connected with compactness in the weak and weak-$*$ topologies.

6.7.1. Theorem. *Let X be a separable normed space. Then every bounded sequence of linear functionals on X contains a weak-$*$ convergent subsequence.*

PROOF. Let $f_n \in X^*$ and $\|f_n\| \leqslant C$. Let us take an everywhere dense countable set $\{x_k\}$ in X. Extract in $\{f_n\}$ a subsequence $\{f_{1,n}\}$ for which the sequence $\{f_{1,n}(x_1)\}$ converges. Next we extract a subsequence $\{f_{2,n}\}$ in $\{f_{1,n}\}$ for which the sequence $\{f_{2,n}(x_2)\}$ converges. Continuing by induction, we construct embedded sequences $\{f_{k,n}\}$ with $k \in \mathbb{N}$ for which the sequences $\{f_{k,n}(x_k)\}$ converge. It is clear that the sequence $\{f_{n,n}\}$ converges at every element x_k. This sequence also converges at every element $x \in X$, since for every $\varepsilon > 0$ there exists a vector x_k such that $\|x - x_k\| \leqslant \varepsilon$, which gives $|f_{n,n}(x) - f_{n,n}(x_k)| \leqslant C\varepsilon$ for all n. It is clear that the equality $f(x) = \lim\limits_{n\to\infty} f_{n,n}(x)$ defines an element of the dual space X^* with $\|f\| \leqslant C$. □

The established property means the *sequential* compactness of the ball in the dual space to any separable normed space with respect to the weak-$*$ topology (in §6.10(ii) we verify the metrizability of this topology on balls). This property does not follows from the usual compactness, and without the separability condition it cannot be guaranteed. For example, the sequence of functionals $f_n(x) = x_n$ on l^∞ does not contain a pointwise convergent subsequence (for every subsequence $\{f_{n_k}\}$ there is an element $x \in l^\infty$ such that $\{x_{n_k}\}$ has no limit). In the general case one has compactness in the weak-$*$ topology. The proof given below employs some facts from the complementary material of Chapter 1. However, for most of applications it is enough to have the previous elementary result.

6.7.2. Theorem. (THE BANACH–ALAOGLU–BOURBAKI THEOREM) *In the space dual to a normed space, closed balls are compact in the weak-$*$ topology.*

PROOF. It suffices to consider the unit ball S of the space X^* dual to a normed space X with the closed unit ball U. According to Tychonoff's theorem, the product of U copies of the closed interval $[-1,1]$ is compact in the product topology, i.e., the space of all functions on U with values in $[-1,1]$ is compact in the topology of pointwise convergence. Let us embed S into $K := [-1,1]^U$ with the aid of the mapping $J(l)(x) = l(x)$, $x \in U$. It is easy to see that this mapping is a homeomorphism between the set S and its image in K. Hence it suffices to verify that $J(S)$ is closed in K. Let y be an element of K that is a limit point for $J(S)$. This means (see 1.9(i)) that there exists a net of elements $y_\alpha \in S$ such that the net $\{y_\alpha(x)\}$ converges to $y(x)$ for every $x \in U$. By the linearity of all y_α convergence holds for every $x \in X$. The function z defined by the formula $z(x) = \lim\limits_{\alpha} y_\alpha(x)$ is linear on X and coincides with y on U. Hence $z \in X^*$. Thus, $y = J(z)$. Of course, a similar reasoning can be given in terms of neighborhoods in the weak-$*$ topology without nets. □

In the case of a Hilbert space H we obtain analogous assertions for the weak topology, since the canonical isomorphism between H and H^* identifies the weak topology on H with the weak-$*$ topology on H^*.

6.7.3. Theorem. *Let H be a Hilbert space. Then every bounded sequence in H contains a weakly convergent subsequence.*

In particular, the closed unit ball in H is sequentially compact in the weak topology.

PROOF. Let $\|h_n\| \leqslant C$. The closure of the linear span of $\{h_n\}$ is a separable Hilbert space. Denote it by H_0. In H_0, according to what has been proved, we can extract a weakly convergent subsequence in $\{h_n\}$. It will converge weakly also in the whole space H, since every functional $l \in H^*$ is represented by a vector $v \in H$, which can be decomposed into the sum $v = v_0 + v'$, where $v' \perp H_0$. Hence the action of l on H_0 coincides with the action of the functional generated by the vector v_0. □

A similar assertion is true for any reflexive Banach space (Exercise 6.10.118). Here we consider the following particular case.

6.7.4. Theorem. *If $1 < p < \infty$, then every bounded sequence in $L^p(\mathbb{R}^n)$ contains a weakly convergent subsequence.*

PROOF. We know that $L^p(\mathbb{R}^n)$ can be identified with the dual to $L^q(\mathbb{R}^n)$, where $q^{-1} + p^{-1} = 1$. Under this identification the weak topology of L^p corresponds to the weak-$*$ topology of the dual space. It remains to apply Theorem 6.7.1. □

Unlike Hilbert spaces (and some other spaces), in the general case the weak topology does not possess the property of the weak-$*$ topology established in Theorem 6.7.1. For example, the sequence of functions $x_n(t) = t^n$ in $C[0,1]$ does not contain a weakly convergent subsequence (although the sequence itself is fundamental in the weak topology). The sequence of functions $x_n(t) = \sin(\pi n t)$ does not even contain a subsequence that would be fundamental in the weak topology (i.e., the values of every continuous linear functional on it would form a Cauchy sequence of numbers). In will be shown in §6.10(iii) that balls in nonreflexive spaces are never weakly compact.

Now we can show that every Banach space is a closed linear subspace in some space $C(K)$, where K is a compact space. Any separable space can be embedded into $C[0,1]$: this fact is proved below in Theorem 6.10.24.

6.7.5. Theorem. *Every Banach space is linearly isometric to a closed linear subspace of the space $C(K)$, where K is a compact space.*

PROOF. For K it is natural to take the closed unit ball of the space X^* with the weak-$*$ topology. Now to every $x \in X$ we associate the function $\psi_x \in C(K)$ by the formula $\psi_x(f) = f(x)$, $f \in K$. Then $\sup_{f \in K} |f(x)| = \|x\|$. Thus, $x \mapsto \psi_x$ is a linear isometry. □

The next result (*Goldstine's theorem*) is yet another example of application of the Hahn–Banach theorem.

6.7.6. Theorem. *Let X be a normed space, U_X and $U_{X^{**}}$ the closed unit balls in X and X^{**}, and let $J\colon X \to X^{**}$ be the canonical embedding. Then the set $J(U_X)$ is everywhere dense in $U_{X^{**}}$ in the topology $\sigma(X^{**}, X^*)$. Hence $J(X)$ is everywhere dense in X^{**} in the topology $\sigma(X^{**}, X^*)$.*

PROOF. By Theorem 6.7.2 the ball $U_{X^{**}}$ is compact in $\sigma(X^{**}, X^*)$. Let V be the closure of $J(U_X)$ in this topology. It is also compact. If $V \neq U_{X^{**}}$, then there is $x^{**} \in U_{X^{**}} \backslash V$. By Corollary 6.3.9 applied to the weak-$*$ topology, there exists an element $l \in X^*$ such that $x^{**}(l) > \sup_{u \in U_X} Ju(l)$. Such a form of the corollary of the Hahn–Banach theorem will be proved in Corollary 8.3.8, so here we give a direct proof of the existence of l. To this end, using the $\sigma(X^{**}, X^*)$-compactness of V, we find a $\sigma(X^{**}, X^*)$-neighborhood of zero W in X^{**} of the form

$$W = \{z^{**} \in X^{**} \colon\ |z^{**}(l_i)| < 1, i = 1, \ldots, n\}, \quad l_i \in X^*,$$

for which $(x^{**} + W) \cap V = \varnothing$. Set $P\colon X^{**} \to \mathbb{R}^n$, $Pz^{**} = \bigl(z^{**}(l_1), \ldots, z^{**}(l_n)\bigr)$. The convex compact set $P(V)$ does not contain the point Px^{**}. Applying the Hahn–Banach theorem to the finite-dimensional space $P(X^{**})$, we obtain a linear functional f on $P(X^{**})$ with $f(Px^{**}) > \sup_{h \in P(U_X)} f(h)$. The functional $f \circ P$ is generated by an element of X^*, since it is a linear combination of $l_1, \ldots, l_n$, because it vanishes on the intersection of the kernels of l_i. Thus, the desired functional $l \in X^*$ is found. It remains to observe that $\|x^{**}\| \leqslant 1$ and $\sup_{u \in U_X} |Ju(l)| = \sup_{u \in U_X} |l(u)| = \|l\|$, so that the inequality $x^{**}(l) > \|l\|$ is impossible. □

With the aid of this theorem it is easy to prove that the reflexivity of a Banach space is equivalent to the weak compactness of its closed unit ball (see Theorem 6.10.10).

6.8. Adjoint and Selfadjoint Operators

Let X, Y be normed spaces. For every operator $T \in \mathcal{L}(X, Y)$ and every functional $y^* \in Y^*$, the function T^*y^* on X given by the formula

$$\langle T^*y^*, x\rangle := \langle y^*, Tx\rangle, \quad \text{i.e.,} \quad T^*y^*(x) := y^*(Tx),$$

is linear and continuous on X. The obtained linear mapping

$$T^*\colon\ Y^* \to X^*$$

is called the *adjoint operator*. It is continuous and satisfies the equality

$$\|T^*\| = \|T\|.$$

Indeed,

$$\|T^*y^*\| = \sup_{\|x\|\leqslant 1} |T^*y^*(x)| = \sup_{\|x\|\leqslant 1} |y^*(Tx)| \leqslant \|T\|\|y^*\|,$$

since $\|Tx\| \leqslant \|T\|$. On the other hand, for every $\varepsilon > 0$ there exists $x \in X$ with $\|x\| = 1$ and $\|Tx\| > \|T\| - \varepsilon$. By the Hahn–Banach theorem there exists a functional $y^* \in Y^*$ with $\|y^*\| = 1$ and

$$|T^*y^*(x)| = |y^*(Tx)| = \|Tx\| > \|T\| - \varepsilon,$$

which gives $\|T^*\| \geqslant \|T\|$.

For all $A, B \in \mathcal{L}(X)$ we have

$$(A + B)^* = A^* + B^*, \quad (AB)^* = B^*A^*, \tag{6.8.1}$$

which is easy to verify.

There is the following connection between the range of the operator and the kernel of its adjoint.

6.8.1. Lemma. *Let $A \in \mathcal{L}(X, Y)$. Then*

$$\overline{A(X)} = \{y \in Y\colon\ f(y) = 0 \quad \forall f \in \operatorname{Ker} A^*\} = \bigcap_{f\in \operatorname{Ker} A^*} \operatorname{Ker} f.$$

PROOF. Let $y = Ax$ and $f \in \operatorname{Ker} A^*$. Then

$$f(y) = f(Ax) = (A^*f)(x) = 0.$$

Thus, $A(X)$ belongs to the right-hand side of the relation in question. Since the latter is closed, it contains the closure of $A(X)$. Conversely, suppose that a vector $y \in Y$ belongs to the right-hand side, but does not belong to $Y_1 := \overline{A(X)}$. By Corollary 6.3.9 there exists $f \in Y^*$ with $f(y) = 1$ and $f|_{Y_1} = 0$. For every $x \in X$ we have $(A^*f)(x) = f(Ax) = 0$. Hence $f \in \operatorname{Ker} A^*$. Then $f(y) = 0$ by our condition on y, which gives a contradiction. □

In the case of a Hilbert space H (real or complex) for an operator $A \in \mathcal{L}(H)$ we define the adjoint operator A^* by the equality

$$(Ax, y) = (x, A^*y).$$

Since the left-hand side is continuous in x, by the Riesz theorem there is a uniquely defined vector A^*y satisfying the indicated equality. It is clear that the operator A^* is linear. The difference from the case of a Banach space is that the adjoint operator acts on the same space as the original one. This definition is consistent with the general case of a Banach space: identifying the functional $l\colon\ x \mapsto (x, v)$ with the vector v, we obtain $(A^*l)(x) = l(Ax) = (Ax, v) = (x, A^*v)$. Note, however, the following nuance arising in the complex case: for a Hilbert space we have $(\lambda A)^* = \overline{\lambda}A^*$, but for a Banach space $(\lambda A)^* = \lambda A^*$. Thus, in case of a complex Hilbert space the adjoint operator in the category of Hilbert spaces does not coincide with the adjoint operator in the category of Banach spaces. This is explained by the fact that the natural isometry between H^* and H is conjugate-linear, but not linear.

In the case of a Hilbert space X we have the obvious equality

$$(A^*)^* = A.$$

Hence here every bounded operator is adjoint to a bounded operator. For general Banach spaces the situation is different, which is discussed in Exercises 6.10.151 and 6.10.152.

The next definition introduces a very important class of operators on complex or real Hilbert spaces.

6.8.2. Definition. *A bounded linear operator A on a Hilbert space H is called selfadjoint if $A^* = A$, i.e., $(Ax, y) = (x, Ay)$ for all $x, y \in H$.*

Sometimes bounded selfadjoint operators are called *Hermitian operators* or symmetric operators, but in case of unbounded not everywhere defined operators one has to distinguish selfadjoint and symmetric operators (see Chapter 10), so in this chapter we do not use the term "symmetric operator".

6.8.3. Example. (i) Let P be the operator of orthogonal projection onto a closed linear subspace H_0 in a Hilbert space H (see §5.4). Then P is selfadjoint. Indeed, $(Px, y) = (Px, Py) = (x, Py)$, because $Px, Py \in H_0$, $x - Px \perp H_0$ and $y - Py \perp H_0$.

(ii) The diagonal operator from Example 6.1.5(vii) is selfadjoint precisely when all α_n are real.

Given a bounded operator A on a real Hilbert space H, one can take the complexification $H_{\mathbb{C}}$ of the space and define the complexification $A_{\mathbb{C}}$ of the operator A by the formula $A_{\mathbb{C}}(x + iy) := Ax + iAy$, $x, y \in H$. We observe that the operator A in a real space is selfadjoint precisely when the operator $A_{\mathbb{C}}$ is selfadjoint. Indeed, if $A = A^*$, then

$$\begin{aligned}\bigl(A_{\mathbb{C}}(x + iy), u + iv\bigr) &= (Ax + iAy, u + iv)\\ &= (Ax, u) + (Ay, v) + i(Ay, u) - i(Ax, v)\\ &= (x, Au) + (y, Av) + i(y, Au) - i(x, Av) = \bigl(x + iy, A_{\mathbb{C}}(u + iv)\bigr).\end{aligned}$$

The complexification of a selfadjoint operator on a real space is actually the direct sum of two copies of this operator. Most of the results of the spectral theory are valid for complex spaces, but in case of selfadjoint operators many facts remain in force in the real case.

In the case of a Hilbert space H, for the adjoint operator in the sense of Hilbert spaces Lemma 6.8.1 can be restated in the following way.

6.8.4. Lemma. *Let $A \in \mathcal{L}(H)$. Then*

$$\overline{A(H)} = (\mathrm{Ker}\, A^*)^{\perp}, \quad \overline{A^*(H)} = (\mathrm{Ker}\, A)^{\perp},$$

moreover, one has the orthogonal decomposition

$$H = \overline{A(H)} \oplus \mathrm{Ker}\, A^* = \overline{A^*(H)} \oplus \mathrm{Ker}\, A.$$

If the operator A is selfadjoint, then $\overline{A(H)} \perp \operatorname{Ker} A$ and

$$H = \overline{A(H)} \oplus \operatorname{Ker} A.$$

PROOF. It is clear from Lemma 6.8.1 that the subspaces $\overline{A(H)}$ and $\operatorname{Ker} A^*$ are mutually orthogonal and $\overline{A(H)} = (\operatorname{Ker} A^*)^{\perp}$, which also gives the orthogonal decomposition of H. Since $A^{**} = A$, we obtain the remaining equalities. □

It is seen from these equalities that the operator A maps $\overline{A^*(H)}$ one-to-one onto $A(H)$, and if the sets $A^*(H)$ and $A(H)$ are closed, then the first one is mapped one-to-one onto the second one.

With the aid of the adjoint operator one can give the following condition for the surjectivity of an operator.

6.8.5. Proposition. *Let X, Y be Banach spaces and $A \in \mathcal{L}(X, Y)$. The equality $A(X) = Y$ is equivalent to the property that for some $c > 0$ we have*

$$\|A^* y^*\|_{X^*} \geqslant c\|y^*\|_{Y^*} \quad \forall\, y^* \in Y^*. \tag{6.8.2}$$

PROOF. Let $A(X) = Y$ and $y^* \in Y^*$. According to Remark 6.2.4 there exists $\varepsilon > 0$ such that for every $y \in Y$ there is $x \in X$ with $Ax = y$ and $\|x\|_X \leqslant \varepsilon^{-1}\|y\|_Y$. Taking y such that $\|y\|_Y = 1$ and $|y^*(y)| \geqslant \|y^*\|_{Y^*}/2$, we obtain

$$\|A^* y^*\|_{X^*}\|x\|_X \geqslant |A^* y^*(x)| = |y^*(Ax)| = |y^*(y)| \geqslant \|y^*\|_{Y^*}/2,$$

whence $\|A^* y^*\|_{X^*} \geqslant \varepsilon\|y^*\|_{Y^*}/2$.

Conversely, suppose we have (6.8.2). In view of Lemma 6.2.1 it suffices to verify that the closure of $A\big(U_X(0,1)\big)$ contains the ball $U_Y(0,c)$. If this is false, then there exists a vector $y \in Y$ with $\|y\|_Y \leqslant c$ not belonging to the indicated closure. According to a corollary of the Hahn–Banach theorem, there is a functional $y^* \in Y^*$ such that $|y^*(y)| > 1$ and $|y^*(Ax)| \leqslant 1$ whenever $\|x\|_X \leqslant 1$. Then $|A^* y^*(x)| = |y^*(Ax)| \leqslant 1$ whenever $\|x\|_X \leqslant 1$, i.e., $\|A^* y^*\|_{X^*} \leqslant 1$. Now from (6.8.2) we have $\|y^*\|_{Y^*} \leqslant c^{-1}$. Since $\|y\|_Y \leqslant c$, we obtain $|y^*(y)| \leqslant 1$, which is a contradiction. □

6.8.6. Corollary. *Let X, Y be Banach spaces and $A \in \mathcal{L}(X, Y)$.* (i) *The set $A(X)$ is closed if and only if the set $A^*(Y^*)$ is closed.*

(ii) *The operator A maps X one-to-one onto Y if and only if A^* maps the space Y^* one-to-one onto X^*.*

PROOF. (i) Let the subspace $Z := A(X)$ be closed. Denote by A_0 the operator A considered with values in Z. According to the proposition above, for some $c > 0$ we have $\|A_0^* z^*\|_{X^*} \geqslant c\|z^*\|_{Z^*}$. Suppose that a sequence $\{y_n^*\} \subset Y^*$ is such that $A^* y_n^* \to x^*$ in X^*. The restriction of y_n^* to Z will be denoted by z_n^*. We observe that $A^* y_n^* = A_0^* z_n^*$, since both functionals give $y_n(Ax)$ on any vector $x \in X$. Therefore, the functionals z_n^* converge in Z^* to some $z^* \in Z^*$. We extend z^* to Y and obtain an element $y^* \in Y^*$. Then $A^* y^* = A_0^* z^* = x^*$, which proves the closedness of $A^*(Y^*)$.

If the closedness of $A^*(Y^*)$ is given, then the subspace $A_0^*(Z^*)$ is also closed. Indeed, if $z_n^* \in Z^*$ and $A_0^* z_n^* \to x^*$, then we can extend z_n^* to functionals $y_n^* \in Y^*$.

As above, we have $A^*y_n^* = A_0^* z_n^*$. By the closedness of $A^*(Y^*)$ there exists $y^* \in Y^*$ with $A^*y_n^* \to A^*y^*$. Let $z^* := y^*|_Z$. Then $A_0^* z^* = A^*y^* = \lim\limits_{n\to\infty} A_0^* z_n^*$. The operator A_0^* is injective, because the set $A(X)$ is dense in Z. Since the operator A_0^* has a closed range, its inverse is continuous. Hence (6.8.2) is true, which gives the equality $A(X) = Z$ and the closedness of $A(X)$.

(ii) If the operator A is an isomorphism, then obviously A^* is also an isomorphism, since for every $x^* \in X^*$ the functional $y^*(y) = x^*(A^{-1}y)$ is continuous and $A^*y^* = x^*$. Moreover, A^* has the zero kernel by Lemma 6.8.1. If A^* is an isomorphism, then by the proposition above $A(X) = Y$. In addition, A has the zero kernel. Indeed, if $Ax = 0$ and $x \neq 0$, then there is $x^* \in X^*$ with $x^*(x) = 1$, which for $y^* = (A^*)^{-1}x^*$ gives the contradictory equality $1 = x^*(x) = A^*y^*(x) = y^*(Ax) = 0$. □

6.8.7. Example. (THE LAX–MILGRAM LEMMA) Let H be a real Hilbert space and let $A \in \mathcal{L}(H)$ be such that $(Ax, x) \geqslant c(x, x)$ for some $c > 0$. Then $A(H) = H$, since $\|A^*y\| \geqslant c\|y\|$ by the estimate $(A^*y, y) = (y, Ay) \geqslant c(y, y)$.

The adjoint operator can be defined for not necessarily bounded linear mappings (see Chapter 10), but such adjoint will not be defined on the whole space.

6.8.8. Example. Let A be a linear mapping from a Banach space X to a Banach space Y such that there exists a linear mapping $A^*\colon Y^* \to X^*$ for which $l(Ax) = (A^*l)(x)$ for all $x \in X$ and $l \in Y^*$. Then A is continuous.

In particular, if X is a Hilbert space and an everywhere defined linear mapping $A\colon X \to X$ is such that for all $x, y \in X$ we have $(Ax, y) = (x, Ay)$, then A is continuous.

PROOF. The graph of A is closed, because if $x_n \to x$ and $Ax_n \to y$, then $l(Ax_n) = (A^*l)(x_n) \to (A^*l)(x) = l(Ax)$ for all $l \in Y^*$, whence $l(y) = l(Ax)$, which means that $Ax = y$. □

6.9. Compact Operators

In this section we begin our study of one special class of operators that is very important for applications.

6.9.1. Definition. *Let X and Y be Banach spaces. A linear operator $K\colon X \to Y$ is called compact if it takes the unit ball to a set with compact closure. The class of compact operators from X to Y is denoted by the symbol $\mathcal{K}(X, Y)$.*

In terms of sequences compactness of the operator K means that for every bounded sequence $\{x_n\}$ in X the sequence $\{Kx_n\}$ must contain a convergent subsequence.

It is clear from the definition that any compact operator is bounded. A similar definition can be introduced in case of not necessarily complete normed spaces, but here there is another, somewhat more general definition (equivalent to the given one in case of complete Y): the image of the unit ball is totally bounded. Such operators are called *completely bounded*.

The simplest example of a compact operator is the zero operator. Another obvious example is a bounded operator with a finite-dimensional range. It is important here that in any finite-dimensional normed space every bounded set is totally bounded. The reader should be warned: not every linear operator with a finite-dimensional range is compact, since there are unbounded finite-dimensional operators (for example, discontinuous linear functionals). The simplest example of an operator that is not compact is the identity mapping of any infinite-dimensional Banach spaces, i.e., the unit operator.

For the sequel we note some elementary properties of totally bounded sets in normed spaces.

6.9.2. Lemma. (i) *A bounded linear operator takes any totally bounded set to a totally bounded set.*

(ii) *If sets A and B in normed spaces X and Y are totally bounded, then $A \times B$ is totally bounded in $X \times Y$.*

(iii) *If sets A and B in a normed space are totally bounded, then the set $\alpha A + \beta B$ is totally bounded for all numbers α and β. If A and B are compact, then this set is compact as well.*

PROOF. The Lipschitzness of a bounded linear operator gives (i) (see Example 1.7.6). Assertion (ii) follows from Exercise 1.9.41. The first assertion in (iii) follows from (i) and (ii), since αA and βB are obviously totally bounded and the operator $(x, y) \mapsto x + y$ from $X \times X$ to X is continuous. The same reasoning gives compactness of $\alpha A + \beta B$ in case of compact A and B. □

The main properties of compact operators are collected in the following theorem.

6.9.3. Theorem. *Let X, Y and Z be Banach spaces.*

(i) *The class $\mathcal{K}(X, Y)$ is a closed linear subspace in the space $\mathcal{L}(X, Y)$.*

(ii) *If $A \in \mathcal{K}(X, Y)$ and $B \in \mathcal{L}(Y, Z)$ or if $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{K}(Y, Z)$, then $BA \in \mathcal{K}(X, Z)$.*

(iii) *An operator $K \in \mathcal{L}(X, Y)$ is compact precisely when the adjoint operator $K^*\colon\ Y^* \to X^*$ is compact.*

PROOF. (i) Let $A, B \in \mathcal{K}(X, Y)$ and let U be the unit ball in X. Then $(A + B)(U) \subset A(U) + B(U)$, $(\lambda A)(U) = \lambda A(U)$. It remains to recall that the algebraic sum of two totally bounded sets and a homothetic image of a totally bounded set are totally bounded (Lemma 6.9.2).

Let $K_n \in \mathcal{K}(X, Y)$, $K \in \mathcal{L}(X, Y)$ and $\|K_n - K\| \to 0$. For every $\varepsilon > 0$ there exists a number N such that $\|K_n - K\| \leqslant \varepsilon$ for all $n \geqslant N$. This means that the set $K_n(U)$ is an ε-net for $K(U)$. Then a finite ε-net existing in $K_n(U)$ serves as a 2ε-net for $K(U)$.

(ii) The set $A(U)$ is totally bounded in Y, so its image under B is totally bounded in Z.

(iii) Let $K \in \mathcal{K}(X, Y)$. Let V be the unit ball in Y^*. Let us verify that the set $K^*(V)$ is totally bounded in X^*. Suppose that we are given a sequence of functionals $f_n \in V$. We have to show that the sequence of functionals $K^* f_n$ contains a subsequence uniformly converging on the unit ball U of the space X. For this we apply the Ascoli–Arzelà theorem (see Theorem 1.8.4). Since $K^*f_n(x) = f_n(Kx)$ and the set $K(U)$ has compact closure, denoted by S, we only need to observe that the functions f_n are uniformly bounded on S and uniformly Lipschitz, which follows from the bound $\|f_n\| \leqslant 1$. Thus, every sequence in $K^*(V)$ contains a convergent subsequence, which means the compactness of $K^*(V)$.

Suppose now that $K^* \in \mathcal{K}(Y^*, X^*)$. It follows from what we have proved that $K^{**}\colon\ X^{**} \to Y^{**}$ is a compact operator. Moreover,

$$K^{**}J_1x = J_2Kx \quad \text{for all } x \in X,$$

where $J_1\colon\ X \to X^{**}$ and $J_2\colon\ Y \to Y^{**}$ are the canonical isometric embeddings. Indeed, for every $f \in Y^*$ we have

$$(K^{**}J_1x)(f) = (J_1x)(K^*f) = (K^*f)(x) = f(Kx) = (J_2Kx)(f).$$

By the isometry of the embedding we obtain the compactness of the closure of the set $K(U)$ in Y. □

Yet another simple property of a compact operator K on any space X is the separability of $K(X)$, which follows obviously from the separability of the images of all balls of radius n (since these images are also totally bounded).

Let us give some examples of compact operators.

6.9.4. Example. (i) Let $\{\alpha_n\}$ be a bounded sequence of numbers. The diagonal operator

$$A\colon\ l^2 \to l^2, \quad (x_n) \mapsto (\alpha_n x_n),$$

is compact precisely when $\lim\limits_{n\to\infty} \alpha_n = 0$.

(ii) Let $\mathcal{K} \in C([0,1]^2)$. Then the *integral operator*

$$Kx(t) = \int_0^1 \mathcal{K}(t,s)x(s)\,ds$$

on the space $C[0,1]$ is compact.

(iii) Let $\mathcal{K} \in L^2([0,1]^2)$. Then the *integral operator*

$$Kx(t) = \int_0^1 \mathcal{K}(t,s)x(s)\,ds$$

on the space $L^2[0,1]$ is compact. The function $\mathcal{K}$ defining the integral operator K is called the *integral kernel*.

(iv) The *Volterra operator*

$$Vx(t) = \int_0^t x(s)\,ds$$

is compact as an operator from $L^1[0,1]$ to $L^p[0,1]$ with $1 \leqslant p < \infty$ and also as an operator from $L^p[0,1]$ to $C[0,1]$ with $p > 1$. However, the operator $V\colon\ L^1[0,1] \to C[0,1]$ is not compact.

PROOF. (i) Let $\lim\limits_{n\to\infty} \alpha_n = 0$. Let us consider the finite-dimensional operators $K_n\colon (x_n) \mapsto (\alpha_1 x_1, \ldots, \alpha_n x_n, 0, 0, \ldots)$. We have $\|K - K_n\| \leqslant \sup_{i>n} |\alpha_i| \to 0$ as $n \to \infty$. Thus, K is a compact operator.

If $\{\alpha_n\}$ contains a subsequence $\{\alpha_{n_i}\}$ such that $|\alpha_{n_i}| \geqslant c > 0$, then the sequence of vectors $Ke_{n_i} = \alpha_{n_i} e_{n_i}$, where e_n is the vector with 1 at the nth place and 0 at all other places, contains no Cauchy subsequence. Hence the operator K is not compact.

(ii) The set M of functions Kx, where $\|x\| \leqslant 1$, is totally bounded by the Ascoli–Arzelà theorem. Indeed, this set is bounded by the boundedness of $\mathcal{K}$. In addition, M is equicontinuous: by the uniform continuity of $\mathcal{K}$, for every $\varepsilon > 0$ there is $\delta > 0$ such that $|\mathcal{K}(t,s) - \mathcal{K}(t',s)| \leqslant \varepsilon$ if $|t - t'| \leqslant \delta$, so

$$|Kx(t) - Kx(t')| \leqslant \int_0^1 |\mathcal{K}(t,s) - \mathcal{K}(t',s)|\,|x(s)|\,ds \leqslant \varepsilon \quad \text{if } \|x\| \leqslant 1.$$

(iii) For every function $x \in L^2[0,1]$ the function $\mathcal{K}(t,s)x(s)$ is integrable in s for almost all t, since by Fubini's theorem $\mathcal{K}(t,\,\cdot\,) \in \mathcal{L}^2[0,1]$ for almost all t. Hence the function Kx is defined almost everywhere. By the Cauchy inequality

$$\left| \int_0^1 \mathcal{K}(t,s)x(s)\,ds \right|^2 \leqslant \int_0^1 |\mathcal{K}(t,s)|^2\,ds \int_0^1 |x(s)|^2\,ds,$$

which after integration in t over $[0,1]$ gives the estimate

$$\int_0^1 |Kx(t)|^2\,dt \leqslant \|x\|^2 \int_0^1 \int_0^1 |\mathcal{K}(t,s)|^2\,ds\,dt.$$

Thus, $\|K\| \leqslant \|\mathcal{K}\|_{L^2([0,1]^2)}$. Now we take a sequence of functions $\mathcal{K}_n$ on $[0,1]^2$ of the form $\mathcal{K}_n(t,s) = \sum_{i,j\leqslant n} c_{ij}\varphi_i(t)\psi_j(s)$, where $\varphi_i, \psi_j \in L^2[0,1]$, such that $\|\mathcal{K}_n - \mathcal{K}\|_{L^2([0,1]^2)}$. The operators K_n defined by the functions $\mathcal{K}_n$ converge in the operator norm to K due to the estimate obtained above. It remains to observe that these operators are finite-dimensional: the range of K_n is contained in the linear span of the functions $\varphi_1, \ldots, \varphi_n$.

(iv) The image of the unit ball U from $L^1[0,1]$ is bounded in $C[0,1]$. If $\{f_n\} \subset U$, then $Vf_n = V(f_n^+) - V(f_n^-)$. The functions $V(f_n^+)$ are monotone and uniformly bounded. Hence one can extract a subsequence pointwise converging on $[0,1]$ (see Exercise 4.5.20). Selecting yet another subsequence $\{f_{n_k}\}$ for which the monotone functions $V(f_{n_k}^-)$ converge everywhere, we obtain a uniformly bounded pointwise convergent sequence $\{Vf_{n_k}\}$. By the Lebesgue dominated convergence theorem it converges in all $L^p[0,1]$. This gives compactness of V as an operator with values in $L^p[0,1]$. But there is no compactness of V with values in $C[0,1]$: the sequence Vf_n, where $f_n = nI_{[0,1/n]}$, is not equicontinuous, since it converges pointwise to the indicator function of $(0,1]$. Finally, for $p > 1$ the operator $V\colon L^p[0,1] \to C[0,1]$ is compact, since in addition to the uniform boundedness of the image of the unit ball from $L^p[0,1]$ one has the equicontinuity of this image, which follows from the estimate

$$|Vx(t) - Vx(t')| \leqslant \left| \int_{t'}^t |x(s)|\,ds \right| \leqslant |t - t'|^{1-1/p},$$

fulfilled by Hölder's inequality. □

Quite often (but not always) for proving the compactness of an operator its approximations by finite-dimensional operators are constructed. In many concrete spaces the class of compact operators coincides with the closure of the set of finite-dimensional operators.

6.9.5. Proposition. *Let H be a Hilbert space. Then the set $\mathcal{K}(H)$ of compact operators coincides with the closure of the set of bounded finite-dimensional operators with respect to the operator norm. If H is separable and $\{e_n\}$ is an orthonormal basis, then for every $K \in \mathcal{K}(H)$ we have $\|K - P_n K\| \to 0$, where P_n is the orthogonal projection onto the linear span of the vectors $e_1, \ldots, e_n$.*

PROOF. We have to show that every operator $K \in \mathcal{K}(H)$ can be approximated by finite-dimensional ones. As noted above, the image of any compact operator is separable, so we can deal with a separable space H and prove the last assertion of the proposition. This assertion follows from the criterion of compactness in H, since the image of the ball under K is contained in a compact set S, whence $\|K - P_n K\|^2 \leqslant \sup_{y \in S} \sum_{i=n+1}^{\infty} |(y, e_i)|^2 \to 0$. □

A similar assertion is true for $C[0,1]$ and more generally for spaces with Schauder bases (see §6.10(iv)). For a long time it was unknown whether this is true for all Banach spaces; only in 1973 did P. Enflo publish a disproving counterexample.

Let us give a simple sufficient condition for the noncompactness of an operator (in case of a Hilbert space this condition is also necessary).

6.9.6. Example. Let X and Y be Banach spaces and $A \in \mathcal{L}(X,Y)$. If $A(X)$ contains an infinite-dimensional closed subspace, then A is not compact. Indeed, if E is a closed subspace in $A(X)$ and U is the unit ball in X, then by Baire's theorem there is $n \in \mathbb{N}$ such that the closure of $A(nU) \cap E$ contains a ball from E. Since this closure is totally bounded, the space E is finite-dimensional.

Note that the image of the closed unit ball under a compact operator can fail to be closed (hence it need not be compact).

6.9.7. Example. (i) Let us take the continuous linear functional l on $C[0,1]$ from Example 6.1.5(iii), which does not attain its maximum on the ball. The image of the closed unit ball under l is the interval $(-1,1)$.

(ii) The image of the closed unit ball in $C[-1,1]$ under the Volterra operator

$$Vx(t) = \int_{-1}^{t} x(s)\, ds$$

is not closed in $C[-1,1]$, since the function $x(t) = |t|$ belongs to the closure of this image, but does not belong to the image itself.

On the other hand, there is a positive result about compactness of the image of the ball.

6.9.8. Proposition. *Let X be a reflexive Banach space* (*for example, a Hilbert space*)*, let Y be a normed space, and let $K\colon X \to Y$ be a completely bounded operator. Then the image of every closed ball in X is compact in the space Y.*

PROOF. Let B be a closed ball and $y_n = Kx_n$, $x_n \in B$. Passing to a subsequence, we can assume that the sequence $\{y_n\}$ is Cauchy. Using the reflexivity of X, we can pass to a subsequence in $\{x_n\}$ that converges weakly to some element $x \in B$ (see Exercise 6.10.118). We assume again that this is the whole original sequence. Then the vectors $y_n = Kx_n$ converge weakly to Kx. It is readily seen (Exercise 6.10.96) that then $\|y_n - Kx\| \to 0$. □

In diverse concrete spaces compact operators may have additional interesting properties. For example, the following *Daugavet theorem* holds.

6.9.9. Example. For every compact operator K on the space $C[0,1]$ one has

$$\|K + \lambda I\| = \|K\| + |\lambda| \quad \forall\, \lambda \in \mathbb{C}.$$

PROOF. It suffices to consider the case $\lambda = 1$. We first consider K of the form $Kx = \sum_{i=1}^n l_i(x)x_i$, where $l_i \in C[0,1]^*$, $x_i \in C[0,1]$. Due to the bound $\|K + \lambda I\| \leqslant \|K\| + |\lambda|$, it suffices to establish the opposite inequality. We prove that for every $\varepsilon > 0$ the inequality $\|K + \lambda I\| \geqslant \|K\| + |\lambda| - \varepsilon$ holds. Let us find a function x for which $\|x\| = 1$ and $\|Kx\| \geqslant \|K\| - \varepsilon/3$. Next we find a point $t \in [0,1]$ such that $\bigl|\sum_{i=1}^n l_i(x)x_i(t)\bigr| = \|Kx\|$. By the Riesz theorem the functionals l_i have the form

$$l_i(x) = \int_{[0,1]} x(s)\, \mu_i(ds),$$

where μ_i are bounded Borel measures on $[0,1]$. We replace the taken point by a point t such that $\mu_i(t) = 0$ for all i (if this was not fulfilled at once) and also $r := \bigl|\sum_{i=1}^n l_i(x)x_i(t)\bigr| \geqslant \|Kx\| - \varepsilon/3$. Let $\sum_{i=1}^n l_i(x)x_i(t) = re^{i\theta}$, $\theta \in \mathbb{R}^1$. We redefine the function x in a small neighborhood of t in order to obtain a continuous function y with $\|y\| = 1$, $y(t) = e^{i\theta}$ and $|l_i(y) - l_i(x)| \leqslant \varepsilon(3n \max_i \|x_i\|)^{-1}$, $i = 1, \ldots, n$ (we assume that not all functions x_i are zero). Then

$$\begin{aligned}
\|Ky + y\| &\geqslant \Bigl|\sum_{i=1}^n l_i(y)x_i(t) + y(t)\Bigr| \\
&\geqslant \Bigl|\sum_{i=1}^n l_i(x)x_i(t) + y(t)\Bigr| - \sum_{i=1}^n |l_i(y) - l_i(x)|\, \|x_i\| \\
&\geqslant |re^{i\theta} + e^{i\theta}| - \varepsilon/3 = r + 1 - \varepsilon/3 \geqslant \|Kx\| + 1 - 2\varepsilon/3 \\
&\geqslant \|K\| + 1 - \varepsilon.
\end{aligned}$$

In the general case we find a sequence of finite-dimensional operators K_n with $\|K - K_n\| \to 0$, which is possible by the compactness of K and Example 6.1.12. Then $\|K_n\| \to \|K\|$ and $\|K_n + I\| \to \|K + I\|$. □

Operators on Hilbert spaces do not possess such a property (it suffices to take a diagonal operator on $\mathbb{C}^2$ with eigenvalues 0 and -1). The equality fails for the noncompact operator $-I$. Werner [**711**] gives a survey on the Daugavet property.

Compact operators are also discussed in Chapter 7.

6.10. Complements and Exercises

6.10(i). Operator ranges and factorization

In this subsection we present a number of useful results connected with the properties of the images of continuous linear mappings and with a close question about the possibility of representing one of the two given operators in the form of the composition of the second operator with some third operator. First we discuss conditions under which a given linear subspace L in a Banach space X coincides with the image of some operator $T \in \mathcal{L}(Z, X)$ on a Banach space Z. If in this situation we are allowed to take arbitrary Banach spaces Z, then it suffices to consider only injective operators T, since $T(Z) = \widetilde{T}(Z/\mathrm{Ker}\,T)$, where $\widetilde{T}$ is the factorization of T by its kernel, i.e., the operator on the quotient space taking the equivalence class $[z]$ of the element $z \in Z$ to Tz. The image $T(Z)$ of an injective operator T can be equipped with the norm

$$\|x\|_T := \|T^{-1}x\|_Z, \quad x \in T(Z).$$

Then $T(Z)$ with this norm turns out to be a Banach space the identity embedding of which into X is continuous. Indeed, we have $\|x\| \leqslant \|T\|\,\|T^{-1}x\|_Z$. If a sequence $\{x_n\} \subset T(Z)$ is Cauchy with respect to the norm above, then the sequence $\{T^{-1}x_n\}$ is Cauchy in Z and hence converges to some element $z \in Z$. Hence we obtain $x_n \to Tz$ and $\|Tz - x_n\|_T \to 0$.

6.10.1. Definition. *A linear subspace E in a Banach space X is called a continuously embedded Banach space if E is equipped with a norm $\|\cdot\|_E$ with respect to which E is complete and the identity mapping $(E, \|\cdot\|_E) \to (E, \|\cdot\|_X)$ is continuous.*

If the balls with respect to the norm $\|\cdot\|_E$ are totally bounded in X, then E is called compactly embedded.

Our discussion above leads to the following conclusion.

6.10.2. Proposition. *A linear subspace L of a Banach space X is the image of some continuous linear operator from a Banach space precisely when L can be equipped with a norm with respect to which it will be a Banach space continuously embedded into X.*

The situation will change if we impose restrictions on Z. In this case it can happen that there are no continuous operators on Z the image of which contains L. For example, there is no continuous operator from l^2 onto l^1: otherwise l^1 would be linearly homeomorphic to a quotient space of l^2, hence to a Hilbert space (and then the space $l^\infty = (l^1)^*$ would be separable). For the same reason $L^2[0,1]$ cannot be mapped onto $C[0,1]$ by means of a bounded operator. However, $C[0,1]$ can be mapped onto $L^2[0,1]$ by means of a bounded operator (Exercise 6.10.174).

Let us note the following property of operator ranges, or, which is the same, continuously embedded Banach spaces.

6.10.3. Proposition. *Let X and Y be Banach spaces and let an operator $A \in \mathcal{L}(X,Y)$ have a nonclosed range. Then the algebraic dimension of the algebraic complement of $A(X)$ in Y is uncountable.*

PROOF. Otherwise there is a finite or countable set of vectors y_n the linear span of which algebraically complements $A(X)$. Denote by E_n the linear span of $y_1, \ldots, y_n$. It is clear that $A(X) + E_n$ is the image of the continuous operator A_n acting from the Banach space $X_n := X \oplus E_n$ to Y by the formula $(x,y) \mapsto Ax+y$. By Baire's theorem for some n the image of the ball of radius n in X_n is dense in some ball in Y. According to Lemma 6.2.1 this gives the equality $A_n(X_n) = Y$. Hence $A(X)$ has a finite codimension. According to Proposition 6.2.12, the image of A is closed, which contradicts the assumption of the proposition. □

Let us mention the following nontrivial result due to V. V. Shevchik, the proof of which can be found in [**704**] (see also Exercise 6.10.165 for the case of Hilbert spaces).

6.10.4. Theorem. *Let X be a separable Banach space and let $E_1 \neq X$ be a Banach space continuously embedded into X and everywhere dense in X. Then there exists a separable Banach space E_2 continuously embedded into X such that E_2 is also everywhere dense and $E_1 \cap E_2 = 0$.*

We now prove a useful result about factorization.

6.10.5. Theorem. *Let X, Y, Z be Banach spaces and let $A\colon X \to Z$ and $B\colon Y \to Z$ be continuous linear operators such that $A(X) \subset B(Y)$. If $\mathrm{Ker}\, B = 0$, then there exists a continuous linear operator $C\colon X \to Y$ such that $A = BC$.*

If the operator B is not injective, then there exists a continuous linear operator $C\colon X \to Y/\mathrm{Ker}\, B$ for which $A = \widetilde{B}C$, where $\widetilde{B}\colon Y/\mathrm{Ker}\, B \to Z$ is the factorization of B by the kernel.

PROOF. If $\mathrm{Ker}\, B = 0$, we have a well-defined linear mapping $C\colon X \to Y$, $Cx := B^{-1}Ax$. This mapping has a closed graph: if $x_n \to x$ and $Cx_n \to y$, then $Ax_n \to Ax$ and $Ax_n = BCx_n \to By$, whence $Ax = By$, i.e., $y = Cx$. Therefore, the operator C is continuous. If the kernel of B is nontrivial, then we pass to the injective operator $\widetilde{B}$. □

6.10.6. Corollary. *If in the situation of the previous theorem the operator B is compact, then A is also compact.*

6.10.7. Example. Since the natural embedding of $C[0,1]$ into $L^2[0,1]$ is not a compact operator (it suffices to consider functions $x_n(t) = \sin(2\pi n t)$), then by the previous corollary there is no compact operator on $L^2[0,1]$ with the range containing $C[0,1]$.

The next result due to Banach and Mazur shows that every separable Banach space is isomorphic to some factor-space of l^1.

6.10.8. Theorem. *For every separable Banach space X, there exists an operator $A \in \mathcal{L}(l^1, X)$ with $A(l^1) = X$.*

PROOF. Let $\{x_n\}$ be a dense sequence in the unit ball of X. The operator

$$A\colon\, l^1 \to X, \quad A\xi = \sum_{n=1}^{\infty} \xi_n x_n,\ \xi = (\xi_n)$$

is well-defined, since the series converges in norm. We have $\|A\xi\| \leqslant \|\xi\|$. The surjectivity of A follows from Lemma 6.2.1, since the image of the unit ball from l^1 is dense in the unit ball from X by construction. □

Note that if X is a Banach space and $A\colon\, l^2 \to X$ is a continuous linear surjection, then X has an equivalent Hilbert norm.

Let us give a typical example of using weak convergence simultaneously in $C[a,b]$ and $L^2[a,b]$ for establishing the compactness of an operator by means of information about its range.

6.10.9. Example. Let $A\colon\, L^2[a,b] \to L^2[a,b]$ be a bounded linear operator such that $A(L^2[a,b]) \subset C[a,b]$. Then the operator A is compact. The same is true if in place of an interval with Lebesgue measure we take an arbitrary topological space T with a bounded Borel measure and replace $C[a,b]$ by $C_b(T)$.

PROOF. We show that for every sequence $\{x_n\}$ bounded in L^2 the sequence $\{Ax_n\}$ contains a subsequence converging in L^2. We know that passing to a subsequence we can assume that $\{x_n\}$ converges weakly to some element x from L^2. By Corollary 6.2.8 the operator A is continuous as an operator with values in the Banach space $C[a,b]$. Hence by weak convergence $x_n \to x$ the sequence $\{Ax_n\}$ is norm bounded and converges weakly in $C[a,b]$ to Ax. Hence for every point $t \in [a,b]$ we have $Ax_n(t) \to Ax(t)$. By the Lebesgue dominated convergence theorem we obtain convergence of Ax_n to Ax with respect to the norm of L^2, as required. It is clear that in this reasoning no special features of the interval are used, so it works for every bounded Borel measure on a topological space. □

Below in Theorem 7.10.27 we establish an even stronger property of operators on L^2 with images in C. We recall that the identity embedding $C[0,1] \to L^2[0,1]$ is not a compact operator.

6.10(ii). Weak compactness in Banach spaces

In a Hilbert space the weak topology coincides with the weak-$*$ topology after identification of the space with its dual. In general Banach spaces (even in dual spaces) there is no this phenomenon. The properties of the weak topology of a Banach space can differ substantially from the properties of the weak-$*$ topology of its dual (but in a reflexive space the weak topology can be identified with the weak-$*$ topology of the dual to its dual). For example, the ball in the weak topology need not be compact. Since weak topologies are often used in applications, we make a short excursion in their theory, which is not included in basic courses. In particular, we prove the most important theorems about weak topologies: the Eberlein–Shmulian and Krein–Shmulian theorems.

First we recall that the closed unit ball of a Banach space is not always compact in the weak topology. We have actually encountered such example: in §6.1 we constructed a continuous linear functional on $C[0,1]$ not attaining its maximum on the closed ball. It turns out that this example exhibits the general picture.

6.10.10. Theorem. *Let X be a Banach space. The following assertions are equivalent*:

(i) *closed balls in X are weakly compact*;

(ii) *every continuous linear functional on X attains its maximum on the closed unit ball*;

(iii) *the space X is reflexive.*

PROOF. Assertion (i) implies (ii), and (iii) implies (i) by the Banach–Alaoglu–Bourbaki theorem, since the weak topology of a reflexive space X can be identified with the weak-$*$ topology of the space X^{**}. The implication (ii)⇒(iii) is a very deep result due to James (see also the next theorem); it is proved, for example, in the book Diestel [**147**, Chapter 1]. We confine ourselves to the proof of the elementary implication (i)⇒(iii). This assertion follows from Theorem 6.7.6, since the weak compactness of the closed unit ball U_X in X gives its $\sigma(X^{**},X^*)$-compactness and hence $\sigma(X^{**},X^*)$-closedness in X^{**}. □

6.10.11. Corollary. (i) *A Banach space X is reflexive precisely when X^* is reflexive.*

(ii) *Closed subspaces of a reflexive Banach space are reflexive.*

PROOF. (i) If X is reflexive, then on X^* the weak topology coincides with the weak-$*$ topology, which by the Banach–Alaoglu–Bourbaki theorem gives the weak compactness of closed balls in X^*. If X^* is reflexive, then by the already proven assertion X^{**} is reflexive. Hence the ball $U_{X^{**}}$ is weakly compact. The ball U_X is closed in $U_{X^{**}}$ in the norm topology and hence is weakly closed. Hence it is weakly compact in X^{**}. Then it is weakly compact in X, since the weak topology X^{**} is stronger than the weak topology of X.

(ii) Let E be a closed subspace of a reflexive Banach space X. Then the unit ball of E is the set $E\cap U_X$, which is weakly compact in X by the weak closedness of E and the weak compactness of U_X. It remains to observe that the weak topology of E is the restriction of the weak topology of X to E. This follows from the fact that every element $l\in E^*$ is the restriction to E of some functional $\widetilde{l}\in X^*$. □

Actually James proved the following even more general fact (see [**147**, p. 19]).

6.10.12. Theorem. *If B is a weakly closed bounded set in a Banach space X, then the weak compactness of B is equivalent to the property that every continuous linear functional attains its maximum on B.*

For working with weak topologies the following Eberlein–Shmulian theorem is extremely important.

6.10.13. Theorem. *Let A be a set in a Banach space X. Then the following conditions are equivalent*:

(i) *the set A has compact closure in the weak topology*;

(ii) *every sequence in A has a subsequence weakly converging in X*;

(iii) *every infinite sequence in A has a limit point in X in the weak topology* (*i.e., a point every neighborhood of which contains infinitely many elements of this sequence*).

In particular, for sets in a Banach space with the weak topology compactness is equivalent to the sequential compactness and is also equivalent to the countable compactness.

PROOF. 1. First we show that any infinite sequence $\{a_n\}$ in a weakly compact set A in a Banach space X contains a weakly convergent subsequence. For this we note the following simple fact (delegated to Exercise 6.10.119): if X^* contains a countable set $\{f_n\}$ separating points in X, then $\bigl(A, \sigma(X, X^*)\bigr)$ is metrizable. Let A_0 be the weak closure of $\{a_n\}$ and let E be the norm closure of the linear span of $\{a_n\}$. Then the set E is weakly closed (being convex) and hence A_0 is weakly compact in E. This enables us to pass to the separable space E. Then E^* contains a countable family of functionals separating points. Hence $\bigl(A_0, \sigma(E, E^*)\bigr)$ is a metrizable compact space. Therefore, there is a subsequence in $\{a_n\}$ that converges weakly in E, hence, as is easily seen, also in X.

2. Suppose now that every infinite sequence in A has a limit point in the weak topology. Let us prove that A is contained in a weakly compact set. We need the following fact: if $Z \subset X^{**}$ is a finite-dimensional subspace, then the unit sphere S_{X^*} in X^* contains a finite set Λ such that

$$\|z^{**}\| \leqslant 2 \max\{|z^{**}(l)| \colon\ l \in \Lambda\} \quad \forall\, z^{**} \in Z.$$

To this end, using the norm compactness of the unit sphere S_Z in Z, we choose a finite 1/4-net $z_1^{**}, \ldots, z_n^{**}$ for S_Z. Next we take elements $l_i \in S_{X^*}$ with $|z_i^{**}(l_i)| > 3/4$. Then for every $z^{**} \in S_Z$ there is an element z_k^{**} with the property $\|z^{**} - z_k^{**}\| < 1/4$, which gives the relation

$$z^{**}(l_k) = z_k^{**}(l_k) + z^{**}(l_k) - z_k^{**}(l_k) > 3/4 - 1/4 = 1/2,$$

whence the desired estimate for all $z^{**} \in Z$ follows.

We observe that A is norm bounded. Otherwise we could find a functional $l \in X^*$ such that $\sup_{a \in A} |l(a)| = \infty$. This would give a sequence $\{a_n\}$ with $|l(a_n)| > n$, hence this sequence cannot have limit points in the weak topology. We shall regard A as a set in X^{**} and denote by B its closure in the topology $\sigma(X^{**}, X^*)$, i.e., the weak-$*$ topology of the space $X^{**} = (X^*)^*$. By the Banach–Alaoglu–Bourbaki theorem B is compact in the indicated topology. Our goal is to show that actually $B \subset X$. Then it will turn out that A is contained in a weakly compact set. Let $x^{**} \in B$ and $l_1 \in S_{X^*}$. The neighborhood $\{y^{**} \in X^{**} \colon\ |(x^{**} - y^{**})(l_1)| < 1\}$ contains an element $a_1 \in A$. Then

$$|(x^{**} - a_1)(l_1)| < 1.$$

Let us consider the linear span Z_1 of the vectors x^{**} and $x^{**} - a_1$. The observation made above gives elements $l_2, \ldots, l_{k_2} \in S_{X^*}$ with the property

$$\|z^{**}\| \leqslant 2 \max\{|z^{**}(l_i)| \colon\ i = 1, \ldots, k_2\} \quad \forall\, z^{**} \in Z_1.$$

Now we take the weak-$*$ neighborhood of x^{**} generated by $l_1, l_2, \ldots, l_{k_2}$ and the number $1/2$ and find an element $a_2 \in A$ in this neighborhood, which gives the bounds

$$|(x^{**} - a_2)(l_i)| < 1/2, \quad i = 1, \ldots, k_2.$$

Next we take the linear span Z_2 of the vectors $x^{**}, x^{**} - a_1, x^{**} - a_2$ and with the aid of our observation find functionals $l_{k_2+1}, \ldots, l_{k_3} \in S_{X^*}$ with the property

$$\|z^{**}\| \leqslant 2\max\{|z^{**}(l_i)|\colon \ i = 1, \ldots, k_3\} \quad \forall\, z^{**} \in Z_2.$$

Continuing this process by induction we obtain a sequence of points $a_n \in A$ and functionals $l_i \in S_{X^*}$, $k_{n-1} < i \leqslant k_n$ for which

$$|(x^{**} - a_n)(l_i)| < 1/2^n, \quad i = 1, \ldots, k_n,$$

$$\|z^{**}\| \leqslant 2\max\{|z^{**}(l_i)|\colon \ i = 1, \ldots, k_{n+1}\} \quad \forall\, z^{**} \in Z_n,$$

where Z_n is the linear span of the elements $x^{**}, x^{**} - a_1, \ldots, x^{**} - a_n$. According to our assumption, the sequence $\{a_n\}$ has a limit point $x \in X$ in the weak topology. Since the norm closed linear span E of the sequence $\{a_n\}$ is weakly closed, we have $x \in E$. In the space X^{**} the element $x^{**} - x$ is a limit point of the sequence $x^{**}, x^{**} - a_1, x^{**} - a_2, \ldots$ in the weak-$*$ topology and hence belongs to the closure F of the linear span of this sequence in the weak-$*$ topology. By our construction

$$\|z^{**}\| \leqslant 2\sup_i |z^{**}(l_i)| \tag{6.10.1}$$

for all z^{**} from the linear span of $x^{**}, x^{**} - a_1, x^{**} - a_2, \ldots$, which obviously extends (6.10.1) to all $z^{**} \in F$. In particular, this inequality is fulfilled for $z^{**} = x^{**} - x$. However, the construction of $\{a_n\}$ and $\{l_i\}$ and the fact that x is a weak limit point of $\{a_n\}$ imply that $|(x^{**} - x)(l_m)| = 0$ for all m. Thus, $\|x^{**} - x\| = 0$, as required. □

6.10.14. Corollary. *A Banach space is reflexive if and only if every separable closed subspace in this space is reflexive.*

PROOF. We have already seen that any closed subspace of a reflexive space is reflexive. Suppose that separable subspaces in a space X are reflexive. Then every sequence $\{x_n\}$ from the unit ball is contained in a separable closed subspace Y, which is reflexive by assumption, hence its closed ball is weakly compact. By the Eberlein–Shmulian theorem $\{x_n\}$ contains a subsequence weakly converging in Y. Then this subsequence also converges weakly in X, so the ball in X is sequentially compact. Applying the Eberlein–Shmulian theorem again, we conclude that the ball in X is weakly compact. An alternative justification follows from the James theorem. □

The next theorem due to Krein and Shmulian is a deep analog of the already known fact (see Proposition 5.5.4) about compactness of the closed convex envelope of a compact set in a Banach space.

6.10.15. Theorem. *Suppose that a set A in a Banach space X is compact in the weak topology. Then the closed convex envelope of A (the intersection of all closed convex sets containing A) is also compact in the weak topology (we recall* *that the closed convex envelope in the norm topology coincides with the closed convex envelope in the weak topology, see Theorem* 6.6.9).

PROOF. We apply Theorem 6.10.12, although there are other proofs. Let V be the closed convex envelope of A with respect to the norm and let $f \in X^*$. By the compactness of A there exists a point $a \in A$ with $f(a) = \sup_{x \in A} f(x)$. We observe that $a \in V$ and $\sup_{x\in V} f(x) = \sup_{x \in A} f(x)$, so the cited theorem applies. □

The Eberlein–Shmulian theorem is very useful for establishing conditions for the weak compactness and weak convergence in concrete spaces. Let us mention a number of typical results, the proofs of which can be found in §4.7(iv) and §4.7(v) in Chapter 4 of [**73**].

6.10.16. Theorem. *Let μ be a finite measure on a measurable space $(\Omega, \mathcal{A})$ and let $\mathcal{F}$ be some set of μ-integrable functions. Then the set $\mathcal{F}$ is uniformly integrable precisely when it has compact closure in the weak topology of $L^1(\mu)$.*

6.10.17. Corollary. *Suppose that $\{f_n\}$ is a uniformly integrable sequence on a space with a finite measure μ. Then there exists a subsequence $\{f_{n_k}\}$ that converges in the weak topology of $L^1(\mu)$ to some function $f \in L^1(\mu)$.*

6.10.18. Corollary. *Let μ be a bounded nonnegative measure and let M be a norm bounded set in $L^1(\mu)$. The closure of M in the weak topology is compact precisely when for every sequence of μ-measurable sets A_n with $A_{n+1} \subset A_n$ and $\bigcap_{n=1}^\infty A_n = \varnothing$ one has*

$$\lim_{n\to\infty} \sup_{f\in M} \int_{A_n} |f|\, d\mu = 0.$$

Let now $(\Omega, \mathcal{A})$ be a measurable space and let $\mathcal{M}(\Omega, \mathcal{A})$ be the Banach space of all real countably additive measures of bounded variation with its natural variation norm $\mu \mapsto \|\mu\|$.

6.10.19. Theorem. *For every set $M \subset \mathcal{M}(\Omega, \mathcal{A})$ the following conditions are equivalent.*

(i) *The closure of M in the topology $\sigma(\mathcal{M}, \mathcal{M}^*)$ is compact.*

(ii) *The set M is bounded in variation and there exists a nonnegative measure $\nu \in \mathcal{M}(\Omega, \mathcal{A})$ (a probability measure if $M \neq \{0\}$) such that the family M is uniformly ν-continuous, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ with the property that*

$$|\mu(A)| \leqslant \varepsilon \quad \textit{for all } \mu \in M \textit{ whenever } A \in \mathcal{A} \textit{ and } \nu(A) \leqslant \delta.$$

In this case the measures from M are absolutely continuous with respect to ν, the closure of the set $\{d\mu/d\nu\colon\ \mu \in M\}$ is compact in the weak topology of $L^1(\nu)$, and for ν one can choose the measure $\sum_{n=1}^\infty c_n|\mu_n|$ for some finite or countable collection $\{\mu_n\} \subset M$ and numbers $c_n > 0$.

(iii) *Every sequence in M contains a subsequence converging on every set from $\mathcal{A}$.*

6.10.20. Corollary. *A sequence of measures $\mu_n \in \mathcal{M}(\Omega, \mathcal{A})$ converges in the topology $\sigma(\mathcal{M}, \mathcal{M}^*)$ if and only if it converges on every set from $\mathcal{A}$. An equivalent condition*:

$$\lim_{n\to\infty} \int_\Omega f(\omega)\, \mu_n(d\omega) = \int_X f(\omega)\, \mu(d\omega) \tag{6.10.2}$$

for every bounded $\mathcal{A}$-measurable function f.

Let us consider an application to passage to the limit in the integral.

6.10.21. Corollary. *Suppose that a sequence of measures $\mu_n \in \mathcal{M}(\Omega, \mathcal{A})$ converges to a measure μ on every set from $\mathcal{A}$ and a sequence of $\mathcal{A}$-measurable functions f_n is uniformly bounded and $\lim\limits_{n\to\infty} f_n(\omega) = f(\omega)$ for every ω. Then*

$$\lim_{n\to\infty} \int_\Omega f_n(\omega)\, \mu_n(d\omega) = \int_X f(\omega)\, \mu(d\omega).$$

PROOF. By Theorem 6.10.19 there exists a probability measure ν on $\mathcal{A}$ such that μ_n and μ are uniformly ν-continuous. Let $|f_n(\omega)| \leqslant C$, $\|\mu_n\| \leqslant C$ and $\varepsilon > 0$. Let us find $\delta > 0$ such that the bound $\nu(A) < \delta$ yields that $|\mu|(A) < \varepsilon$ and $|\mu_n|(A) < \varepsilon$ for all n. By the Egorov theorem there exists a set A with $\nu(A) > 1 - \delta$ on which convergence $f_n \to f$ is uniform. Hence there is $N \in \mathbb{N}$ such that $|f_n(\omega) - f(\omega)| \leqslant \varepsilon$ if $\omega \in A$ and $n \geqslant N$. It remains to observe that

$$\left| \int_\Omega (f_n - f)\, d\mu_n \right| \leqslant \int_A |f_n - f|\, d|\mu_n| + \int_{\Omega\backslash A} |f_n - f|\, d|\mu_n| \leqslant C\varepsilon + 2C\varepsilon$$

and that the integrals of f with respect to the measures μ_n converge to the integral of f with respect to the measure μ. □

In the case where, for example, $\Omega = [0,1]$ and $\mathcal{A}$ is the Borel σ-algebra, the space $\mathcal{M}(\Omega, \mathcal{A})$ coincides with the dual to $C[0,1]$ by the Riesz theorem, so on $\mathcal{M}(\Omega, \mathcal{A})$ there is also the weak-$*$ topology. It should not be confused with the weak topology $\sigma(\mathcal{M}, \mathcal{M}^*)$. Convergence of measures μ_n to μ in the weak-$*$ topology means the equality (6.10.2) for every continuous function f, while convergence in the topology $\sigma(\mathcal{M}, \mathcal{M}^*)$ is (6.10.2) for all bounded Borel functions f. For example, the Dirac measures $\delta_{1/n}$ at the points $1/n$ tend to the Dirac measure δ_0 at zero in the weak-$*$ topology of the space of measures, but not in the weak topology of the Banach space $C[0,1]^*$. This circumstance becomes especially important, because in many applications (in particular, in probability theory and in the theory of random processes) the weak-$*$ topology of the space of measures is traditionally called merely the weak topology (see, for example, [**73**, Chapter 8]). Corollary 6.10.21 is false for the weak-$*$ convergence of measure. It suffices to take the measures $\delta_{1/n}$ and the functions f_n defined as follows: $0 \leqslant f_n \leqslant 1$, $f_n(1/n) = 1$, $f_n(t) = 0$ if $|t - 1/n| > 1/(2n)$.

Let us give several additional results and remarks in connection with the weak and weak-$*$ topologies.

6.10.22. Proposition. *Let E be a normed space.*

(i) *The space E with the weak topology is metrizable precisely when E is finite-dimensional.*

(ii) *The unit ball U_E of the space E with the weak topology is metrizable precisely when E^* is separable.*

PROOF. (i) If the topology $\sigma(E, E^*)$ is metrizable by a metric d and E is infinite-dimensional, then the ball of radius n^{-1} centered at zero contains a weakly open set, which is unbounded and hence contains vectors x_n with $\|x_n\| \geqslant n$. Then $d(x_n, 0) \to 0$, but $\{x_n\}$ cannot converge weakly to zero being unbounded, which is a contradiction.

(ii) If E^* is separable, then one can take a countable everywhere dense set $\{f_n\}$ in the unit ball of E^*. It is readily seen that the metric

$$d(x,y) := \textstyle\sum_{n=1}^{\infty} 2^{-n}|f_n(x-y)|$$

generates the weak topology on the ball U_E. Conversely, suppose that the ball U_E is metrizable in the weak topology. Let us take a countable collection of neighborhoods of zero $U_{f_{n,1},\ldots,f_{n,k_n}}$ the intersections of which with the ball U_E give a basis of neighborhoods of zero in U_E. Let us show that the linear span of $f_{n,i}$ is dense in E^*. Let Y be the closure of this linear span. If $f \in E^*\backslash Y$, then by the Hahn–Banach theorem there exists an element $x^{**} \in E^{**}$ with $\|x^{**}\| = 1$, $x^{**}|_Y = 0$ and $x^{**}(f) = c > 0$. The weak neighborhood of zero $V := \{x \in U_E\colon\ |f(x)| < c/2\}$ in U_E contains some set $U_{f_{n,1},\ldots,f_{n,k_n}} \cap U_E$. By Theorem 6.7.6 there is a vector $x \in U_E$ for which $|x^{**}(f) - f(x)| < c/2$ and $|x^{**}(f_{n,i}) - f_{n,i}(x)| < 1$ for each $i = 1, \ldots, k_n$. Since $x^{**}(f_{n,i}) = 0$, we have $|f_{n,i}(x)| < 1$, i.e., $x \in U_{f_{n,1},\ldots,f_{n,k_n}} \cap U_E$. However, $|f(x)| > c/2$, because $x^{**}(f) = c$. Therefore, $x \notin V$, which is a contradiction. □

The dual to an infinite-dimensional Banach space cannot be metrizable in the weak-$*$ topology, but the dual to an incomplete infinite-dimensional normed space can be metrizable in the weak-$*$ topology (see Exercise 6.10.148).

6.10.23. Theorem. *Let X be a separable normed space. Then the closed unit ball U^* of the space X^* with the weak-$*$ topology is metrizable and compact. For a metric generating the weak-$*$ topology on the ball one can take*

$$d(f,g) := \textstyle\sum_{n=1}^{\infty} 2^{-n}|f(x_n) - g(x_n)|,$$

where $\{x_n\}$ is a sequence dense in the unit ball from X. In particular, U^* contains a countable set separating points of X.

PROOF. This assertion is a special case of the assertion from Exercise 1.9.68, but it can be derived directly from the facts proved above. To this end we first observe that from Theorem 6.7.1 one can easily deduce the compactness of the ball in X^* with the metric d (the fact that d is a metric is obvious). The identity mapping from the ball with the metric d to the ball with the weak-$*$ topology is continuous, since convergence $f_n \to f$ in the metric d on the ball yields the weak-$*$ convergence. Hence by Theorem 1.7.9 this mapping is a homeomorphism. Of course, it is not difficult to verify directly the continuity of the identity mapping from the weak-$*$ topology to the metric. □

The next remarkable theorem due to Banach and Mazur establishes some universality of the space $C[0,1]$ in the category of separable Banach spaces.

6.10.24. Theorem. *Every separable Banach space is linearly isometric to some closed linear subspace in* $C[0,1]$.

PROOF. In case of separable X the space K from the proof of Theorem 6.7.5 (the unit ball in X^* with the weak-$*$ topology) is a compact metric space, as shown above. According to Exercise 1.9.67, the compact space K is homeomorphic to a compact set in $[0,1]^\infty$. Hence we can assume that $K\subset[0,1]^\infty$. Proposition 1.9.24 gives a continuous surjection π of the Cantor set in $[0,1]$, denoted here by K_1, onto $[0,1]^\infty$. Let $K_0:=\pi^{-1}(K)$. The mapping $f\mapsto f\circ\pi$ is a linear isometry of the space $C(K)$ to $C(K_0)$, since

$$\sup_{t\in K_0}\bigl|f\bigl(\pi(t)\bigr)\bigr| = \sup_{x\in[0,1]^\infty}|f(x)|.$$

It remains to embed $C(K_0)$ into $C[0,1]$ by a linear isometry. For this we extend every function from $C(K_0)$ to a continuous function on $[0,1]$ by a linear interpolation on every interval constituting the complement of K_0 in $[0,1]$. □

Thus, separable normed spaces can be thought as linear subspaces in $C[0,1]$.

6.10.25. Remark. From the Banach–Steinhaus theorem we know that if a sequence of continuous linear functionals f_n on a Banach space X is such that for every $x\in X$ the sequence $\{f_n(x)\}$ converges, then there is an element $f\in X^*$ to which $\{f_n\}$ converges in the weak-$*$ topology. Thus, the space X^* is sequentially complete in the weak-$*$ topology. One can ask an analogous question about the weak topology in X. Suppose that a sequence of vectors $x_n\in X$ is such that for every $f\in X^*$ the sequence $\{f_n(x)\}$ converges. Is it true that $\{x_n\}$ converges weakly to some vector $x\in X$? Generally speaking, this is false. For example, in the space c_0 the sequence of vectors $x_n=(1,\ldots,1,0,0,\ldots)$, where 1 is at the first n positions, has no weak limit, but for every element $f\in c_0^*=l^1$ the sequence $\{f(x_n)\}$ converges. A space X in which convergence of $\{f(x_n)\}$ for every $f\in X^*$ yields weak convergence of $\{x_n\}$ is called *weakly sequentially complete*. It follows from what has been said at the beginning of this remark that every reflexive Banach space is weakly sequentially complete, since its weak topology can be identified with the weak-$*$ topology of this space regarded as the dual to its dual. For example, a Hilbert space is weakly sequentially complete. There exist non-reflexive spaces with this property: for example, l^1 and $L^1(\mu)$.

Let us give without proof (which can be found in [**75**, Theorem 3.8.15] or [**533**, Chapter IV, §6.4]) the following deep result due to Krein and Shmulian.

6.10.26. Theorem. *Let X be a Banach space and let $V\subset X^*$ be a convex set. If the intersection of V with every closed ball of radius n centered at zero is closed in the topology $\sigma(X^*,X)$, then V is also closed in the topology $\sigma(X^*,X)$.*

If X is separable, then for the closedness of V in the topology $\sigma(X^*,X)$ is suffices that V contain the limits of all its weak-$*$ convergent sequences.

See Exercise 8.6.74 in Chapter 8 about some topology on X^* connected with this theorem.

6.10.27. Corollary. *Let X be a Banach space and let F be a linear function on X^*. The following conditions are equivalent*:

(i) *the function F is continuous in the topology $\sigma(X^*, X)$*;

(ii) *there exists $x \in X$ with $F(l) = l(x)$ for all $l \in X^*$*;

(iii) *the restriction of F to the unit ball U_{X^*} in X^* is continuous in the topology $\sigma(X^*, X)$*;

(iv) *the set $F^{-1}(0) \cap U_{X^*}$ is closed in the topology $\sigma(X^*, X)$.*

Finally, if X is separable, then this is also equivalent to the property that $\lim_{n\to\infty} F(l_n) = 0$ for every sequence $\{l_n\} \subset X^*$ that is weak-$*$ convergent to zero.

A Banach space X is called a space with the *Dunford–Pettis property* if convergence $x_n \to 0$ in the weak topology of X and convergence $l_n \to 0$ in the topology $\sigma(X^*, X^{**})$ in the space X^* implies convergence $l_n(x_n) \to 0$. For example, an infinite-dimensional Hilbert space does not possess this property: it suffices to take for $x_n = l_n$ an orthonormal sequence. The space c_0 has the Dunford–Pettis property, since weak convergence in $l^1 = (c_0)^*$ yields convergence in norm (Exercise 6.10.104). Let us give a less obvious example.

6.10.28. Example. The space $C[0, 1]$ possesses the Dunford–Pettis property.

PROOF. Since $C[0, 1]^*$ is the space of bounded Borel measures on $[0, 1]$, we can apply Corollary 6.10.21. □

One more example of a space with the Dunford–Pettis property is $L^1[0, 1]$ (Exercise 6.10.178).

One should bear in mind that although the weak topology of an infinite-dimensional Banach space is always strictly weaker than its norm topology (Example 6.6.3), it can happen that the supplies of convergent sequences in the weak topology and the norm topology coincide. This is the case in l^1 (Exercise 6.10.104). The situation is different with the weak-$*$ convergence, as the following Josefson–Nissenzweig theorem shows (its proof can be found in Diestel [**148**, Chapter XII]).

6.10.29. Theorem. *If X is an infinite-dimensional Banach space , then there exists a sequence of functionals $f_n \in X^*$ with $\|f_n\| = 1$ that is weak-$*$ convergent to zero. Moreover, every functional $f \in X^*$ with $\|f\| \leqslant 1$ is the limit of some sequence of elements $f_n \in X^*$ of unit norm converging in the weak-$*$ topology.*

Let us mention a couple of interesting results connected with l^1. Proofs and references can be found in Albiac, Kalton [**9**]. The first result is due to H. Rosental.

6.10.30. Theorem. *Let $\{x_n\}$ be a bounded sequence in an infinite-dimensional Banach space X. Then it contains either a weakly fundamental subsequence or a subsequence $\{x_{n_k}\}$ such that the mapping $T\colon\ l^1 \to X$, $(\xi_k) \mapsto \sum_{k=1}^\infty \xi_k x_{n_k}$, is a homeomorphism from l^1 onto the closed subspace generated by $\{x_{n_k}\}$.*

The next result is due to E. Odell and H. Rosental.

6.10.31. Theorem. *A separable Banach space X has no closed subspaces isomorphic to l^1 precisely when every element $x^{**} \in X^{**}$ is the limit of some sequence $\{x_n\} \subset X$ in the topology $\sigma(X^{**}, X^*)$.*

All concrete infinite-dimensional Banach spaces we have encountered so far possess the property that they contain subspaces isomorphic to some of the simplest spaces l^p with $1 \leqslant p < \infty$ or c_0. For a long time it was an open question whether there exist infinite-dimensional spaces not containing l^p and c_0. Finally, in 1974 such example was constructed by a Soviet mathematician B. S. Tsirelson.

6.10(iii). The Banach–Saks property and uniform convexity

We shall say that a Banach space X possesses the *Banach–Saks property* if every norm bounded sequence $\{x_n\}$ in X contains a subsequence $\{x_{n_k}\}$ such that the sequence of arithmetic means

$$\frac{x_{n_1} + \cdots + x_{n_k}}{k}$$

converges in norm.

A normed space E with a norm $\|\cdot\|$ is called *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if} \quad \|x\| = 1,\ \|y\| = 1 \quad \text{and} \quad \left\|\frac{x+y}{2}\right\| \geqslant 1 - \delta, \quad \text{then} \quad \|x - y\| \leqslant \varepsilon.$$

The spaces $L^p(\mu)$ with $1 < p < \infty$ are uniformly convex (for a proof, see §4.7(iii) in [**73**]).

6.10.32. Theorem. *All uniformly convex Banach spaces possess the Banach–Saks property.*

For a proof, see Diestel [**147**, Chapter 3, §7]. For example, the space $L^p(\mu)$ with $1 < p < \infty$ possesses the Banach–Saks property. The validity of this property for a Hilbert space can be easily verified directly.

6.10.33. Example. Hilbert spaces possess the Banach–Saks property.

PROOF. Passing to a subsequence we can assume that $\{x_n\}$ converges weakly to some vector x. In addition, we can assume that $x = 0$. Set $n_1 = 1$. Since $(x_{n_1}, x_n) \to 0$, there exists a number $n_2 > n_1$ with $|(x_{n_1}, x_{n_2})| \leqslant 1$. If numbers $n_1 < n_2 < \cdots < n_k$ are already chosen, we find a number $n_{k+1} > n_k$ such that

$$|(x_{n_j}, x_{n_{k+1}})| \leqslant k^{-1}, \quad j = 1, \ldots, k.$$

This is obviously possible by weak convergence of $\{x_n\}$ to zero. We observe that $\sup_n \|x_n\| = M < \infty$. Hence

$$\frac{\|x_{n_1} + \cdots + x_{n_k}\|^2}{k^2} \leqslant \frac{kM^2 + 2 \cdot 1 + \cdots + 2(k-1)(k-1)^{-1}}{k^2} \leqslant \frac{M^2 + 2}{k},$$

which shows norm convergence of the arithmetic means. □

The space $L^1[0,1]$ does not have the Banach–Saks property, which is obvious from consideration of the functions $nI_{[0,1/n]}$.

6.10.34. Theorem. *Any space with the Banach–Saks property is reflexive.*

PROOF. We show that every continuous linear functional f on a space E with the Banach–Saks property attains its maximum on the closed unit ball U. Let us find $u_n \in U$ such that $f(u_n) \to \|f\|$. Passing to a subsequence, we can assume that the elements $s_n := (u_1 + \cdots + u_n)/n$ converge in norm to some $u \in U$. It is clear that $f(s_n) \to \|f\|$. Hence $f(u) = \|f\|$. □

Any uniformly convex space X possesses the following property: if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $\|x - x_n\| \to 0$. Indeed, we can assume that $\|x_n\| = \|x\| = 1$. Then $\|x_n + x\|/2 \to 1$, since if $\|x + x_n\|/2 \leqslant q < 1$, then $|l(x + x_n)/2| \leqslant q$ whenever $\|l\| \leqslant 1$. Hence $|l(x)| \leqslant q$, which yields that $\|x\| \leqslant q$, a contradiction. However, the indicated property is weaker than the uniform convexity. In this connection we mention the following theorem due to Kadec and Klee (see Diestel [**147**, Chapter IV, §4]).

6.10.35. Theorem. *Let X be a Banach space with the separable dual X^*. Then X possesses an equivalent norm that is Fréchet differentiable outside the origin and generates the norm $\|\cdot\|_*$ on X^* with the following property: if $f_n \to f$ in the weak-$*$ topology and $\|f_n\|_* \to \|f\|_*$, then $\|f_n - f\|_* \to 0$.*

The proof of the following interesting result can be read in Fabian, Habala, Hájek, Montesinos Santalucía, Pelant, Zizler [**185**, p. 259]; see also Exercise 6.10.172 for the Hilbert case.

6.10.36. Theorem. *Suppose that the norm of a Banach space X is Fréchet differentiable outside the origin. Then every bounded closed convex set in X is some intersection of closed balls. In particular, this is true for Hilbert spaces.*

6.10(iv). Bases, approximations and complements

As we have seen above, the most important attribute of Hilbert spaces are orthogonal bases. Many Banach space possess topological bases.

6.10.37. Definition. *Let X be a separable Banach space. A sequence $\{h_n\}$ in X is called a Schauder basis or a topological basis if for every $x \in X$ there is a unique sequence of numbers $\{c_n(x)\}$ such that $x = \sum_{n=1}^{\infty} c_n(x)h_n$, where the series converges in norm.*

It is clear that a Schauder basis is a linearly independent set. Note that in an infinite-dimensional Banach space a topological basis cannot be an algebraic basis (a Hamel basis), since the latter is always uncountable.

By the uniqueness of expansions the functionals $l_n\colon x \mapsto c_n(x)$ are linear. It turns out that they are automatically continuous! Note that $l_i(h_j) = \delta_{ij}$ also by the uniqueness of expansions.

6.10.38. Proposition. *All functionals l_i are continuous. Hence the finite-dimensional mappings $P_n\colon x \mapsto \sum_{i=1}^{n} l_i(x)h_i$ are continuous. In addition, the norm $\|x\|_\infty := \sup_n \|P_n x\|$ is equivalent to the original norm.*

PROOF. Since $\|P_n x\| \to \|x\|$, we have $\|x\| \leqslant \|x\|_\infty$. Let us show that X is complete with the norm $\|\cdot\|_\infty$. Then by the Banach inverse mapping theorem we obtain the equivalence of both norms. This will give the boundedness of all mappings P_n with respect to the original norm and the estimate $\sup_n \|P_n\| < \infty$. This yields the continuity of all l_n, since $P_n x - P_{n-1}x = l_n(x)h_n$. Suppose that a sequence $\{x_j\}$ is Cauchy with respect to the new norm. Then it is Cauchy with respect to the original norm and hence converges in it to some $x \in X$. We have to show that $\|x - x_j\|_\infty \to 0$. We observe that for every fixed n the sequence of vectors $P_n x_j$ is also Cauchy with respect to the new norm and hence converges in the original norm to some vector $y_n \in X$. The whole sequence $\{P_n x_j\}$ is contained in the finite-dimensional space X_n generated by $h_1, \ldots, h_n$. On finite-dimensional subspaces the functionals l_i are continuous, so for every $i = 1, \ldots, n$ there exists a limit

$$l_i(y_n) = \lim_{j\to\infty} l_i(P_n x_j) = \lim_{j\to\infty} l_i(x_j) =: c_i,$$

independent of n by the second equality. Let us verify the equality $x = \sum_{i=1}^\infty c_i h_i$ with respect to the original norm. For a given number $\varepsilon > 0$ we find n such that $\|x_n - x_m\|_\infty \leqslant \varepsilon$ for all $m \geqslant n$. Let us take k_0 such that $\|x_n - P_k x_n\| \leqslant \varepsilon$ for all $k \geqslant k_0$. For such k we have

$$\begin{aligned}
\|y_k - x\| &= \lim_{m\to\infty} \|P_k x_m - x_m\| \\
&\leqslant \limsup_{m\to\infty} \Big[\|P_k x_m - P_k x_n\| + \|P_k x_n - x_n\| + \|x_n - x_m\| \Big] \\
&\leqslant \limsup_{m\to\infty} \Big[\|x_m - x_n\|_\infty + \varepsilon + \|x_n - x_m\|_\infty \Big] \leqslant 3\varepsilon.
\end{aligned}$$

Hence $\|y_k - x\| \to 0$. By the uniqueness of expansions we have the equality $y_k = P_k x$. Therefore,

$$\begin{aligned}
\|x_n - x\|_\infty &= \sup_{k\geqslant 1} \|P_k x_n - P_k x\| \leqslant \limsup_{m\to\infty} \sup_{k\geqslant 1} \|P_k x_n - P_k x_m\| \\
&= \limsup_{m\to\infty} \|x_n - x_m\|_\infty \to 0
\end{aligned}$$

as $n \to \infty$. Thus, X is complete with the new norm. □

With respect to the new norm $\|\cdot\|_\infty$ the projections P_n have unit norms (which is not always true for the original norm).

Schauder bases were constructed in many concrete Banach spaces. For example, the Haar functions χ_n defined by the formulas $\chi_1(t) = 1$,

$$\chi_{2^k+l}(t) = \begin{cases} 1 & \text{if } t \in [(2l-2)2^{-k-1}, (2l-1)2^{-k-1}], \\ -1 & \text{if } t \in \big((2l-1)2^{-k-1}, (2l)2^{-k-1}\big], \\ 0 & \text{else}, \end{cases}$$

where $k = 0, 1, 2, \ldots$ and $l = 1, 2, \ldots, 2^k$, form a Schauder basis in $L^p[0,1]$ if $1 \leqslant p < \infty$. The Faber–Schauder functions

$$\varphi_0(t) = 1, \quad \varphi_n(t) = \int_0^t \chi_n(s)\, ds, \quad n \geqslant 1,$$

form a Schauder basis in $C[0,1]$ (these functions were introduced in 1910 by H. Faber, who discovered that they form a basis). The function $\varphi_0 = 1$ is constant, the function $\varphi_1(t) = t$ is linear, and the subsequent functions φ_n have graphs that are equilateral triangles of height 1 and bases of the form $[2^{-k}, 2^{1-k}]$, vanishing outside these bases. The partial sum with number n of the expansion of f with respect to the functions φ_i is the result of the linear interpolation between the values of f at the points $0, 2^{1-n}, 2^{2-n}, \ldots, 1$ for $n \geqslant 1$. For example, we have $S_0 f(t) = c_0\varphi_0(t)$, where $c_0 = f(0)$, $S_1 f(t) = c_0\varphi_0(t) + c_1\varphi_1(t)$, where $c_1 = f(1) - f(0)$, next, $S_2 f(t) = c_0\varphi_0(t) + c_1\varphi_1(t) + c_2\varphi_2$, where $c_2 = f(1/2) - f(0)/2 - f(1)/2$ and so on. From this one can readily derive the uniform convergence of $S_n f$ to f.

In some spaces the attempts to construct a basis failed for a long time. For example, in 1974 S.V. Bochkarev solved the problem posed by Banach and constructed a Schauder basis in the space of functions analytic in the unit disc and continuous on the closed disc (equipped with the sup-norm). The problem of existence of a Schauder basis in every separable Banach space remained open for several decades. This was one of the most famous problems in the theory of Banach spaces. Finally, in 1973 a Swedish mathematician P. Enflo published his celebrated counterexample. Simultaneously he solved negatively another very difficult old problem about the existence of a separable Banach spaces without the approximation property.

A Banach space X is said to possesses the *approximation property* if for every compact set $K \subset X$ and every $\varepsilon > 0$ there exists a continuous finite-dimensional operator T such that $\|x - Tx\| < \varepsilon$ for all $x \in K$. It is known that this is equivalent to the property that, for every Banach space Z, the set of finite-dimensional operators is dense with respect to the operator norm in the space $\mathcal{K}(Z,X)$ of compact operators. If X has a Schauder basis $\{h_n\}$, then the projections $x \mapsto \sum_{i=1}^n x_i h_i$ converge to the identity operator uniformly on compact sets, hence X has the approximation property. However, there exist spaces with the approximation property, but without Schauder bases. It is now known that spaces without the approximation property (hence without Schauder bases) exist even among closed subspaces of c_0 and l^p with $p > 2$ (they certainly also exist among closed subspaces of the universal space $C[0,1]$).

If a linearly independent sequence $\{h_n\}$ is a Schauder basis of the closure of its linear span, then it is called a *basic sequence*. This is equivalent to the property that for some $C > 0$ for all $n < m$ and all scalars $\alpha_1, \ldots, \alpha_m$ we have $\big\|\sum_{i=1}^n \alpha_i h_i\big\| \leqslant C\big\|\sum_{i=1}^m \alpha_i h_i\big\|$. Indeed, if $\{h_n\}$ is a Schauder basis of the closure Y of its linear span, then the indicated estimate with $C = 1$ is true for the equivalent norm $\|\cdot\|_\infty$. Conversely, suppose that this estimate holds. Then, for any convergent series $\sum_{i=1}^\infty \alpha_i h_i$ with the zero sum it follows that all α_i are zero. If this series converges to some x, then we set $x_i := \alpha_i$. It remains to observe that the set of all vectors x representable in this form is a closed subspace. This is easily seen from the estimate $\big\|\sum_{i=1}^n \alpha_i h_i\big\| \leqslant C\|x\|$ following from our condition as $m \to \infty$. Hence this closed subspace is Y.

Basic sequences exist in all spaces. The next fact was already known in Banach's time.

6.10.39. Theorem. *In every infinite-dimensional Banach space there is an infinite basic sequence.*

PROOF. We employ the following fact: let Y be a finite-dimensional subspace of an infinite-dimensional Banach space X. Then for every $\varepsilon > 0$ there exists a vector $h \in X$ with $\|h\| = 1$ such that $\|y\| \leqslant (1+\varepsilon)\|y + \lambda h\|$ for all $y \in Y$ and $\lambda \in \mathbb{R}^1$. Since we have $y/\lambda \in Y$, this inequality can be restated as follows: $\|y\| \leqslant (1+\varepsilon)\|y+h\|$ for all $y \in Y$, or, equivalently, $(1+\varepsilon)\|y+h\| \geqslant 1$ for all vectors $y \in Y$ of unit norm. For the proof we assume that $\varepsilon \in (0,1)$ and $y_1, \ldots, y_m$ is an $(\varepsilon/2)$-net in the unit sphere of Y. Pick functionals $f_i \in X^*$ with $\|f_i\| = 1$ and $f_i(y_i) = 1$. Since X is infinite-dimensional, there is a vector h of unit norm such that $f_i(h) = 0$ for each i. Let $y \in Y$ and $\|y\| = 1$. There is y_i with $\|y - y_i\| < \varepsilon/2$. Then for every number λ we have

$$\|y + \lambda h\| \geqslant \|y_i + \lambda h\| - \frac{\varepsilon}{2} \geqslant f_i(y_i + \lambda h) - \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2} \geqslant (1+\varepsilon)^{-1},$$

as required.

We now take $\varepsilon > 0$ and numbers $\varepsilon_n > 0$ such that $\prod_{n=1}^{\infty}(1+\varepsilon_n) < 1+\varepsilon$. Let $\|h_1\| = 1$. By induction we obtain vectors h_n with

$$\|y\| \leqslant (1+\varepsilon_n)\|y + \lambda h_{n+1}\|$$

for all y from the linear span of $h_1, \ldots, h_n$ and all $\lambda \in \mathbb{R}^1$. It is readily seen that $\{h_n\}$ is a basic sequence and $\|P_n\| < 1+\varepsilon$ for all n. □

Not every linearly independent sequence with a dense linear span in a Banach space is a Schauder basis.

6.10.40. Example. The functions $1, \sin nt, \cos nt$ do not form a Schauder basis in the space $C_{2\pi}$ of 2π-periodic functions, which is clear from the existence of a function in $C_{2\pi}$ with the Fourier series divergent at zero. The functions $x_n(t) = t^n$ do not form a Schauder basis in $C[0,1]$ and $L^2[0,1]$. For the proof we observe that the expansion $x = \sum_{n=1}^{\infty} c_n(x)x_n$ in $C[0,1]$ or in $L^2[0,1]$ ensures the real analyticity of the function x on $[0,1)$, since we have convergence of the series of the integrals, i.e., the series $\sum_{n=1}^{\infty} c_n(x)(n+1)^{-1}$, which yields convergence of the power series $\sum_{n=1}^{\infty} c_n(x)t^n$ if $|t| < 1$.

Nevertheless, one can construct a Schauder basis in $C[0,1]$ consisting of polynomials. As was shown by H. Faber back in 1914, in $C[0,1]$ there is no Schauder basis consisting of polynomials h_n of degree n. In 1990 A. A. Privalov proved that if polynomials h_n form a Schauder basis in $C[0,1]$, then for some $\varepsilon > 0$ and for all sufficiently large n one has $\deg h_n \geqslant (1+\varepsilon)n$. In addition, for every $\varepsilon > 0$ there is a Schauder basis $\{h_n\}$ with $\deg h_n \leqslant (1+\varepsilon)n$. It was shown by M. A. Skopina in 2001 that such polynomials can be chosen even orthogonal (for references, see Odinets, Yakubson [**461**]).

The next result due to Krein–Milman–Rutman shows some stability of bases.

6.10.41. Proposition. *Let $\{u_n\}$ be a Schauder basis in a Banach space X, $\|u_n\| = 1$ and $\sup_n \|P_n\| = K$, where P_n is the projection on the linear span of $u_1, \ldots, u_n$. If a sequence $\{v_n\} \subset X$ is such that $\sum_{n=1}^{\infty} \|u_n - v_n\| < (2K)^{-1}$, then $\{v_n\}$ is also a Schauder basis.*

PROOF. For any $x = \sum_{n=1}^{\infty} x_n u_n$, let us set $Tx = \sum_{n=1}^{\infty} x_n v_n$. Since $|x_n| = \|P_n x - P_{n-1}x\| \leqslant 2K\|x\|$, the series converges by convergence of the series of $x_n u_n$ and $x_n(v_n - u_n)$, moreover,

$$\|x - Tx\| \leqslant \sum_{n=1}^{\infty} |x_n| \, \|u_n - v_n\| \leqslant q\|x\|, \quad q = 2K \sum_{n=1}^{\infty} \|u_n - v_n\| < 1.$$

Hence $\|I - T\| < 1$. So T is invertible (see Theorem 7.1.3), whence our assertion follows. □

6.10.42. Corollary. *If a Banach space has a Schauder basis, then such a basis can be picked in every everywhere dense set.*

In some problems the lack of a Schauder basis is compensated by biorthogonal systems and Markushevich bases.

6.10.43. Definition. *Let X be a nonzero Banach space. A pair of sequences $\{x_n\} \subset X$ and $\{l_n\} \subset X^*$ is called biorthogonal if $l_i(x_j) = \delta_{ij}$.*

If, in addition, the linear span of $\{x_n\}$ is dense in X and the functionals l_n separate points in X, then $\{x_n\}$ is called a Markushevich basis in X.

Note that if the pair of sequences $\{x_n\} \subset X$ and $\{l_n\} \subset X^*$ is biorthogonal, then the sequence $\{x_n\}$ is *minimal* in the following sense: no element x_n belongs to the closure of the linear span of the remaining elements x_k, $k \neq n$ (otherwise we obtain $l_n(x_n) = 0$). For $X = 0$ zero is a basis.

The next result was obtained by A. I. Markushevich.

6.10.44. Theorem. *Every separable Banach space has a Markushevich basis. Moreover, such a basis can be found in every dense linear subspace.*

PROOF. Let us embed the given space X as a closed subspace in $C[0,1]$ and take a linearly independent sequence $\{y_n\}$ whose linear span is dense in X. Let $\{x_n\}$ be the result of othogonalization of $\{y_n\}$ in $L^2[0,1]$. Set $l_n(x) = (x, x_n)_{L^2}$, $x \in X$. If $l_n(x) = 0$ for some $x \in X$ for all n, then the element x is orthogonal to the closed linear span of $\{x_n\}$ in $L^2[0,1]$ and hence is orthogonal to X in $L^2[0,1]$. Therefore, $(x,x)_{L^2} = 0$. Thus, $x = 0$. □

A simple example of a Markushevich basis that is not a Schauder basis is the system of functions $\exp(int)$, $n \in \mathbb{Z}$, in the complex space $C[0, 2\pi]$ of continuous functions with $x(0) = x(2\pi)$ equipped with the sup-norm. In the general case it can happen that $\sup_n \|x_n\| \, \|l_n\| = \infty$. However, it is known that for every $\varepsilon > 0$ one can find a Markushevich basis with $\sup_n \|x_n\| \, \|l_n\| \leqslant 1 + \varepsilon$. The question whether this is true for $\varepsilon = 0$ remains open.

In finite-dimensional spaces there are biorthogonal systems (*Auerbach systems*) with the following property.

6.10.45. Proposition. *In a Banach space X of finite dimension n one can find vectors $x_1, \ldots, x_n$ and linear functionals $l_1, \ldots, l_n$ such that $\|x_i\| = \|l_i\| = 1$ and $l_i(x_j) = \delta_{ij}$.*

PROOF. We can assume that $X = \mathbb{R}^n$ with some norm. Let B be the closed unit ball in this norm. For every collection of vectors $y_1, \ldots, y_n \in B$ we denote by $V(y_1, \ldots, y_n)$ the determinant of the matrix $(y_i, y_j)_{i,j\leqslant n}$, where $y_i = (y_{i,1}, \ldots, y_{i,n})$. The function V attains its maximum on B at some collection of vectors $x_1, \ldots, x_n \in B$. It is clear that $\|x_1\| = \cdots = \|x_n\| = 1$. Set

$$l_i(x) := V(x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)/V(x_1, \ldots, x_n).$$

Then $l_i(x_i) = 1$, $|l_i(x)| \leqslant 1$ for all $x \in B$, i.e., $\|l_i\| = 1$. Finally, $l_i(x_j) = 0$ if $i \neq j$, since $V(y_1, \ldots, y_n) = 0$ if there are two equal vectors among y_i. □

Yet another important geometric property in Banach spaces is connected with the existence of bounded projections. We shall say that a closed subspace E of a Banach space X is *complemented* in X if there exists a closed subspace $D \subset X$ such that $X = E \oplus D$. This gives a bounded operator $P\colon\ X \to E$ with $P(X) = E$ and $Px = x$ for all $x \in E$, i.e., a bounded projection onto E. Conversely, the existence of such projection gives a complement to E in the form $D = P^{-1}(0)$. As we know, in a Hilbert space every closed subspace has the orthogonal complement and hence is complemented. It turns out that there are no other spaces with such a property: J. Lindenstrauss and L. Zafriri proved that if in a Banach space X every closed subspace is complemented, then X is linearly homeomorphic to a Hilbert space. Let us give an interesting concrete example of a subspace that is not complemented.

6.10.46. Example. The space $C[0,1]$ has a closed subspace E linearly isometric to $L^2[0,1]$. This subspace is not complemented.

PROOF. We know that $C[0,1]$ possesses the Dunford–Pettis property (Example 6.10.28), but E does not. Let us show that if E were complemented, then it would also have the Dunford–Pettis property. Let P be a bounded projection onto E. If $x_n \to 0$ in the weak topology of E, then the same is true also for the weak topology of $C[0,1]$. Let $l_n \in E^*$ be such that $l_n \to 0$ in the topology $\sigma(E^*, E^{**})$. Set $f_n = l_n \circ P$. Then $f_n \in C[0,1]^*$. Let $F \in C[0,1]^{**}$. Set $G(l) := F(l \circ P)$, $l \in E^{**}$. Clearly, $G \in E^{**}$. Then $F(f_n) = G(l_n) \to 0$ by our assumption. Therefore, $f_n(x_n) \to 0$, i.e., $l_n(x_n) \to 0$, which gives the Dunford–Pettis property in E. Thus, we have obtained a contradiction. □

The space c_0 also has no complement in l^∞ (Exercise 6.10.117).

Let us prove an interesting result due to A. Sobszyk on extensions of pointwise converging sequences of functionals.

6.10.47. Proposition. *Let X be a separable normed space, let $Y \subset X$ be a linear subspace, and let $\{f_n\} \subset Y^*$ be such that $\lim\limits_{n\to\infty} f_n(y) = 0$ for all $y \in Y$*

and $\|f_n\| \leqslant 1$. *Then there exist functionals* $\widetilde{f}_n \in X^*$ *with* $\widetilde{f}_n|_Y = f_n$, $\|\widetilde{f}_n\| \leqslant 2$ *and* $\lim\limits_{n\to\infty} \widetilde{f}_n(x) = 0$ *for all* $x \in X$.

PROOF. Let $\{x_i\}$ be a countable everywhere dense set in the unit ball of X with $\{x_i\} \cap Y = \varnothing$. It suffices to construct extensions with norms at most 2 pointwise converging to zero on every vector x_i.

This can be easily done if we prove the following fact: for every fixed k and every $\varepsilon > 0$ there exists $N(k, \varepsilon)$ such that for all $n \geqslant N(k, \varepsilon)$ the functionals f_n have extensions $g_n \in X^*$ such that $\|g_n\| \leqslant 2$ and $|g_n(x_i)| \leqslant \varepsilon$, $i = 1, \ldots, k$. Let L denote the linear span of Y and $x_1, \ldots, x_k$. If the aforementioned fact is false, then $\{f_n\}$ contains a subsequence $\{f_{n_j}\}$ for which all f_{n_j} have no extensions with the indicated properties. We can assume that this is the whole original sequence. We know that f_n has an extension $\varphi_n \in X^*$ with $\|\varphi_n\| \leqslant 1$. The sequence of vectors $\bigl(g_n(x_1), \ldots, g_n(x_k)\bigr) \in \mathbb{R}^k$ is bounded and hence has a convergent subsequence. Again we can assume that the whole sequence converges. Then the limit $l(x) := \lim\limits_{n\to\infty} g_n(x)$ exists for all $x \in L$. It is clear that l is a linear functional on L and $\|l\| \leqslant 1$. In addition, $l|_Y = 0$. We extend l to a functional $l_0 \in X^*$ with $\|l_0\| \leqslant 1$. Set $\psi_n := f_n - l_0$. Then $\psi_n|_Y = f_n$, $\|\psi_n\| \leqslant 2$ and for all sufficiently large n we have $|\psi_n(x_i)| \leqslant \varepsilon$, $i = 1, \ldots, k$, which contradicts our supposition and completes the proof. $\square$

This result can be restated as follows.

6.10.48. Corollary. *Let X be a separable normed space, let $Y \subset X$ be a linear subspace, and let $T\colon\ Y \to c_0$ be a continuous operator. Then T extends to an operator $S\colon\ X \to c_0$ such that $\|S\| \leqslant 2\|T\|$.*

Therefore, c_0 is complemented in every separable Banach space isometrically containing it.

The separability is important here: as we have already noted, c_0 is not complemented in l^∞ (Exercise 6.10.117). Up to an isomorphism c_0 is the unique separable Banach space that is complemented in every separable Banach space into which it is embedded as a closed subspace (this result is due to M. Zippin, see [**293**]).

6.10(v). Operators on ordered vector spaces

In the theory of ordered vector spaces, briefly touched upon in §5.6(iii), an important role is played by positive functionals and operators. Let E be an ordered vector space. We say that a linear functional f on E is *positive* (and write $f \geqslant 0$) if $f(x) \geqslant 0$ whenever $x \geqslant 0$. This is equivalent to the property that $f(x) \leqslant f(y)$ whenever $x \leqslant y$. Similarly one defines positive operators between ordered spaces.

For positive functionals there are some analogs of the Hahn–Banach theorem.

6.10.49. Theorem. *Let p be a positively homogeneous convex functional on an ordered vector space E. If on a linear subspace $E_0 \subset E$ we are given a linear functional f satisfying the condition*

$$f(x) \leqslant p(x+z) \quad \textit{for all } x \in E_0 \textit{ and all } z \geqslant 0, \tag{6.10.3}$$

then f extends to a linear functional on all of E satisfying the indicated condition for all $x \in E$.

PROOF. Set $q(x) = \inf\{p(x+z)\colon\ z \geqslant 0\}$, $x \in E$. Since

$$0 = f(0) \leqslant p(0+z) = p(z) \quad \text{if } z \geqslant 0,$$

for any $z \geqslant 0$ we have $p(x+z)+p(-x) \geqslant p(z) \geqslant 0$, i.e., $q(x) \leqslant -p(-x) > -\infty$. It is readily seen that $q(\lambda x) = \lambda q(x)$ for all $x \in E$, $\lambda \geqslant 0$. The inequality $q(x+y) \leqslant q(x) + q(y)$ follows from the fact that for every $\varepsilon > 0$ there exist $z_1, z_2 \geqslant 0$ such that $q(x) > p(x+z_1) - \varepsilon$ and $q(y) > p(y+z_2) - \varepsilon$. Indeed, the latter gives

$$q(x)+q(y) > p(x+z_1)+p(y+z_2)-2\varepsilon \geqslant p(x+y+z_1+z_2)-2\varepsilon \geqslant q(x+y)-2\varepsilon,$$

since $z_1 + z_2 \geqslant 0$. By assumption $f \leqslant q$ on E_0. It remains to apply the usual Hahn–Banach theorem to f and q. □

6.10.50. Corollary. *Condition* (6.10.3) *is necessary and sufficient for the existence of a linear extension $\widetilde{f}$ of the function f to E such that $\widetilde{f} \geqslant 0$ and $\widetilde{f} \leqslant p$ on E. In particular, a linear functional f on a linear subspace E_0 has a positive linear extension to E precisely when there exists a positively homogeneous convex function p satisfying condition* (6.10.3).

PROOF. If this condition is fulfilled, then the extension constructed in the theorem is nonnegative, since for all $z \geqslant 0$ we have

$$-\widetilde{f}(z) = \widetilde{f}(-z) \leqslant p(-z) = 0.$$

Conversely, if such an extension $\widetilde{f}$ exists, then we have

$$f(x) = \widetilde{f}(x) \leqslant \widetilde{f}(x) + \widetilde{f}(z) = \widetilde{f}(x+z) \leqslant p(x+z) \quad \text{for all } x \in E_0 \text{ and } z \geqslant 0,$$

i.e., (6.10.3) is fulfilled. Finally, if f extends to a positive linear functional $\widetilde{f}$ on E, then (6.10.3) is fulfilled with $p = \widetilde{f}$. □

Let us consider some simple examples of positive linear functionals without positive extensions.

6.10.51. Example. (i) Let E be the space of all bounded real functions on the real line with its natural partial order and let E_0 be the linear span of indicator functions of bounded intervals. The functional f on E_0 defined as the Riemann integral is linear and positive. If it had an extension to a positive linear functional $\widetilde{f}$ on E, then we would have $\alpha := \widetilde{f}(1) \in \mathbb{R}^1$. Since $I_J \leqslant 1$ for every interval J, we obtain $\widetilde{f}(1 - I_J) \geqslant 0$, whence $\widetilde{f}(I_J) \leqslant \alpha$, which is impossible if the length of J is greater than α.

(ii) Let $E = \mathbb{R}^\infty$ be the space of all real sequences with the partial order defined by the coordinate-wise comparison and let $E = c$ be the subspace of all sequences with a finite limit. Then the functional $f(x) := \lim\limits_{n\to\infty} x_n$ has no positive linear extensions to E. Indeed, if such an extension $\widetilde{f}$ exists, then the element x with $x_n = n$ is mapped to some number $\alpha \in [0, +\infty)$. Let us take a natural number $k > \alpha$ and an element y with $y_n = k$ for all $n \geqslant k$ and $y_j = 0$ if $j < k$. Then $k = \widetilde{f}(y) \leqslant \widetilde{f}(x) = \alpha$, which is a contradiction.

Let us give two positive results.

6.10.52. Corollary. *Let f be a positive linear functional on a linear subspace $E_0 \subset E$. Suppose that for every $x \in E$ there exists $y \in E_0$ such that $x \leqslant y$. Then f extends to a positive linear functional on E. In particular, this is true if there is a point in the algebraic kernel of the positive cone of E contained in E_0.*

PROOF. Set $U := \{x \in E_0\colon\ f(x) < 1\}$ and

$$W := \{w \in E\colon\ w \leqslant u \text{ for some } u \in U\}.$$

We observe that the set W is convex and $0 \in W$. In addition, for every $x \in E$ there exists $\varepsilon > 0$ such that $\varepsilon x \in W$. Indeed, by condition there exists $y \in E_0$ with $x \leqslant y$. One can take $\varepsilon > 0$ such that $f(\varepsilon y) < 1$. Then $\varepsilon x \in W$. Let us take for p the Minkowski functional of the set W. It follows from what has been said above that p is a positively homogeneous convex function. Let us verify condition (6.10.3). If it fails, then there exist elements $x \in E_0$ and $z \geqslant 0$ such that $f(x) > 1$ and $p(x+z) < 1$. Then for some $\lambda \in (0,1)$ we have $(x+z)/\lambda \in W$, i.e., $(x+z)/\lambda \leqslant u$, where $u \in U$. Hence $x + z \leqslant \lambda u$ for some $x \leqslant \lambda u$. Hence $f(x) \leqslant \lambda f(u) < 1$, which is a contradiction. Applying Theorem 6.10.49 we complete the proof of the first assertion. If the positive cone has a nonempty algebraic kernel intersecting E_0, then we take a point $z_0 \in E_0$ in this kernel. Then for every $x \in E$ there exists $\alpha > 0$ with $z_0 - \alpha x \geqslant 0$, i.e., $x \leqslant z_0/\alpha \in E_0$. □

6.10.53. Corollary. *Let z_0 belong to the algebraic kernel of the positive cone K. Then the existence of a positive linear functional f with $f(z_0) = 1$ is equivalent to the condition $-z_0 \notin K$.*

PROOF. Let $-z_0 \notin K$. On the one-dimensional space generated by z_0 we define f by the formula $f(\lambda z_0) = \lambda$. If $\lambda z_0 \in K$, then by condition we have $\lambda \geqslant 0$ and hence $f(\lambda z_0) \geqslant 0$. Now we can apply the previous corollary. If f exists, then in case $-z_0 \in K$ we would obtain $f(-z_0) \geqslant 0$, i.e., $f(z_0) \leqslant 0$ contrary to the equality $f(z_0) = 1$. □

Note that if an ordered vector space E is equipped with a norm (or a topology making it a topological vector space) and the positive cone has inner points in the corresponding topology, then every positive linear functional is automatically continuous, being bounded from below on an nonempty open set.

6.10.54. Example. On the spaces $L^\infty(\mu)$ and $C_b(T)$, where T is a topological space, all positive linear functionals are continuous.

A simple example of a discontinuous positive linear functional on an ordered normed space is the function $f(\varphi) = \varphi(0)$ on the space of all polynomials on $[0,1]$ (or on the space of all continuous functions) equipped with the norm from $L^2[0,1]$.

However, the existence of inner points in the positive cone is not necessary for the automatic continuity of all positive linear functionals on a given ordered normed space. For the definition of a Banach lattice, see §5.6(iii).

6.10.55. Proposition. *Every positive linear functional f on a Banach lattice E is automatically continuous. In addition,*

$$\|f\| = \sup\{|f(x)|\colon\ x \geqslant 0,\ \|x\| = 1\}.$$

In particular, this is true for $L^p(\mu)$, $1 \leqslant p \leqslant \infty$. Here, in the case, for example, of Lebesgue measure μ the positive cone in $L^p(\mu)$ with $p < \infty$ has no inner points.

A similar assertion is true for positive operators on E with values in a normed lattice.

PROOF. Let f be a discontinuous positive functional. Then there exists a sequence of elements $x_n \in E$ such that $\|x_n\| \leqslant 2^{-n}$ and $f(x_n) \geqslant n$. We can assume that $x_n \geqslant 0$, since we have $|f(x)| \leqslant |f(|x|)|$. Let $x = \sum_{n=1}^{\infty} x_n$. Then $f(x) \geqslant f(x_n) \geqslant n$ for all n, which is impossible. The assertion about $\|f\|$ follows from the estimate $|f(x)| \leqslant |f(|x|)|$. The case of a positive operator is completely analogous. If the measure μ is not concentrated at finitely many atoms, for example, is Lebesgue measure on an interval, then every neighborhood of every function from $L^p(\mu)$ with $p < \infty$ contains a function that is strictly negative on a set of a nonzero measure. Hence the positive cone has no inner points. □

It follows from this proposition that a positive functional $\varphi \mapsto \varphi(0)$ on the linear subspace of continuous functions in the space $L^2[0,1]$ has no positive extensions to $L^2[0,1]$.

Let us mention the Kantorovich theorem on extensions of positive operators, a particular case of which is Corollary 6.10.52 (for a proof, see, e.g., [**14**, p. 29]).

6.10.56. Theorem. *Let X be an ordered vector space with a positive cone K, let $X_0 \subset X$ be a linear subspace such that $X_0 + K = X$, and let Y be a complete vector lattice. Then every positive linear operator $T\colon\ X_0 \to Y$ extends to a positive linear operator on all of X.*

6.10.57. Proposition. *Let E be an ordered vector space, let f be a linear functional on E, and let p be a positively homogeneous convex functional. Suppose also that on a linear subspace $E_0 \subset E$ we are given a linear functional g. Then the following conditions are equivalent*:

(i) *there is a linear extension $\widetilde{g}$ of the functional g to E such that*

$$f(z) \leqslant \widetilde{g}(z) \quad \textit{for all } z \geqslant 0 \textit{ and } \quad \widetilde{g}(x) \leqslant p(x) \quad \textit{for all } x \in E; \qquad (6.10.4)$$

(ii) *there is a linear extension $\widetilde{g}$ of the functional g to E such that*

$$\widetilde{g}(x) + f(z) \leqslant p(x + z) \quad \textit{for all } z \geqslant 0 \textit{ and } x \in E; \qquad (6.10.5)$$

(iii) *the functional g satisfies the condition*

$$g(y) + f(z) \leqslant p(y + z) \quad \textit{for all } z \geqslant 0 \textit{ and } y \in E_0. \qquad (6.10.6)$$

PROOF. Condition (i) implies (ii), because we have

$$p(x+z) \geqslant \widetilde{g}(x+z) = \widetilde{g}(x) + \widetilde{g}(z) \geqslant \widetilde{g}(x) + f(z).$$

Condition (ii) implies (iii). Let (iii) be fulfilled. Set $p_1(x) := p(x) - f(x)$ if $x \in E$ and $g_1(y) := g(y) - f(y)$ if $y \in E_0$. Then by (6.10.5) we obtain

$g_1(y) \leqslant p_1(y+z)$ if $y \in E_0$, $z \geqslant 0$. By Theorem 6.10.49 there is a linear extension $\widetilde{g}_1$ of the functional g_1 to E with $\widetilde{g}_1(x) \leqslant p_1(x+z)$ for all $x \in E$, $z \geqslant 0$. Set $\widetilde{g} := \widetilde{g}_1 + f$. Then $\widetilde{g}_1$ is a linear extension of g. For all $z \geqslant 0$ and $x \in E$ we have $\widetilde{g}(x) + f(z) = \widetilde{g}_1(x) + f(x) + f(z) \leqslant p(x+z)$. For $x = -z$ we obtain $f(z) \leqslant g(z)$ and for $z = 0$ we have $\widetilde{g}(x) \leqslant p(x)$. □

6.10.58. Corollary. *Let E be an ordered vector space, let f be a linear functional on E, and let p be a positively homogeneous convex functional. Then a necessary and sufficient condition for the existence of a linear functional g on E such that $g(x) \leqslant p(x)$ for $x \in E$ and $f(z) \leqslant g(z)$ for all $z \geqslant 0$ is this: $f(z) \leqslant p(z)$ for all vectors $z \geqslant 0$.*

PROOF. The necessity is clear. For the proof of sufficiency we take $E_0 = \{0\}$ with $g = 0$ on E_0. Then condition (6.10.6) is fulfilled, so g extends linearly to E such that (6.10.4) is fulfilled. □

We now turn to ordered normed spaces. Suppose that an ordered vector space E is equipped with a norm $\|\cdot\|$ such that for some $c > 0$ we have

$$\|x\| \leqslant c\|z\| \quad \text{whenever } 0 \leqslant x \leqslant z. \qquad (6.10.7)$$

Then the positive cone is called *normal*. For example, in the Banach spaces $B(\Omega)$ and $L^p(\mu)$ (more generally, in all Banach lattices) this condition is fulfilled, but in the space $C^1[0,1]$ of continuously differentiable functions with its natural norm and the pointwise partial order this condition fails. First we note the following fact which can be easily derived from Theorem 6.10.49.

6.10.59. Proposition. *If E_0 is a Banach sublattice of a Banach lattice E, then every positive functional on E_0 extends to a positive functional on E with the same norm.*

Let us discuss decompositions of functionals in differences of positive functionals.

6.10.60. Proposition. *Let E be an ordered normed space with a normal positive cone satisfying condition* (6.10.7). *Then for every continuous linear functional f there exist positive continuous functionals f_1 and f_2 such that*

$$f = f_1 - f_2, \quad \|f_1\| \leqslant c\|f\|, \quad \|f_2\| \leqslant (1+c)\|f\|.$$

Conversely, if this is true for every $f \in E^*$, then the positive cone is normal and (6.10.7) *holds with $2(1+c)$ in place of c.*

PROOF. Let $\|f\| = 1$, $U := \{x \in E\colon \|x\| < 1/c\}$, $W := U - K$, where $K = \{z \in E\colon z \geqslant 0\}$. It is readily seen that W is an open convex set and $0 \in W$. For all $z \in K \cap W$ we have $\|z\| \leqslant 1$, since $0 \leqslant z = u - z_1$, where $\|u\| < 1/c$ and $z_1 \geqslant 0$, whence $0 \leqslant z \leqslant u$. By (6.10.7) this gives $\|z\| \leqslant c\|u\| \leqslant 1$. Denote by p the Minkowski functional of the set W. We observe that

$$f(z) \leqslant p(z) \text{ and } p(-z) < -1 \text{ for all } z \in K.$$

Indeed, setting $a := p(z)$, for each $\varepsilon > 0$ we have $p\bigl(z/(a+\varepsilon)\bigr) < 1$, hence $(a+\varepsilon)^{-1}z \in W \cap K$. Then $\|z/(a+\varepsilon)\| < 1$ as shown above. Hence we have

$f(z) \leqslant \|z\| \leqslant a$. Next, $-z \leqslant 0 \in U$, whence $-z \in W$ and hence $p(-z) < 1$. The previous corollary gives a linear functional $g \leqslant p$ such that $f(z) \leqslant g(z)$ for all $z \in K$. The functional g is positive, since for all $z \in K$ and $\lambda > 0$ we have $g(-\lambda z) \leqslant p(-\lambda z) < 1$, whence $g(z) \geqslant -1/\lambda$. Finally, if $\|x\| < 1$, we obtain $x/c \in U \subset W$, which gives $g(x/c) \leqslant p(x/c) \leqslant 1$. Thus, $\|g\| \leqslant c$. Now we can let $f_1 := g$, $f_2 := g - f$.

Suppose now that for every $f \in E^*$ there exist positive functionals f_1 and f_2 with $f = f_1 - f_2$ and $\|f_1\| \leqslant (1+c)\|f\|$, $\|f_2\| \leqslant (1+c)\|f\|$. Let $0 \leqslant x \leqslant z$. A corollary of the Hahn–Banach theorem gives $f \in E^*$ with $f(x) = \|x\|$ and $\|f\| = 1$ (we can assume that $x \neq 0$). Taking f_1 and f_2 as above, we obtain $0 \leqslant f_i(x) \leqslant f_i(z) \leqslant (1+c)\|z\|$. Hence $\|x\| = f(x) = f_1(x) - f_2(x)$ does not exceed $2(1+c)\|z\|$. □

The following result about decompositions of functionals in differences of positive functionals gives a bit more than Proposition 6.10.60.

6.10.61. Theorem. *Let $\mathcal{F}$ be a vector lattice of bounded functions on a set Ω containing 1. Suppose that on $\mathcal{F}$ we are given a linear functional L continuous with respect to the norm $\|f\| = \sup_\Omega |f(x)|$. Then L can be represented in the form $L = L^+ - L^-$, where $L^+ \geqslant 0$, $L^- \geqslant 0$ and for all nonnegative $f \in \mathcal{F}$ we have*

$$L^+(f) = \sup_{0 \leqslant g \leqslant f} L(g), \quad L^-(f) = -\inf_{0 \leqslant g \leqslant f} L(g). \tag{6.10.8}$$

In addition, letting $|L| := L^+ + L^-$, for all $f \geqslant 0$ we have

$$|L|(f) = \sup_{0 \leqslant |g| \leqslant f} |L(g)|, \quad \|L\| = L^+(1) + L^-(1).$$

A similar assertion with the exception of the equality is true for all continuous linear functionals on normed vector lattices.

PROOF. For nonnegative functions $f, g \in \mathcal{F}$ and every function $h \in \mathcal{F}$ such that $0 \leqslant h \leqslant f + g$ we can write $h = h_1 + h_2$, where h_1, $h_2 \in \mathcal{F}$, $0 \leqslant h_1 \leqslant f$, $0 \leqslant h_2 \leqslant g$. Indeed, let $h_1 = \min(f, h)$, $h_2 = h - h_1$. Then h_1, $h_2 \in \mathcal{F}$, $0 \leqslant h_1 \leqslant f$ and $h_2 \geqslant 0$. Finally, $h_2 \leqslant g$. Indeed, if $h_1(x) = h(x)$, then $h_2(x) = 0$ if $h_1(x) = f(x)$, then $h_2(x) = h(x) - f(x) \leqslant g(x)$, since $h \leqslant g + f$.

Let L^+ be defined by equality (6.10.8). We observe that the quantity $L^+(f)$ is finite, since $|L(h)| \leqslant \|L\| \, \|h\| \leqslant \|L\| \, \|f\|$. It is clear that $L^+(tf) = tL^+(f)$ for all nonnegative numbers t and $f \geqslant 0$. Let $f \geqslant 0$ and $g \geqslant 0$ belong to $\mathcal{F}$. Using the notation above, we obtain

$$\begin{aligned} L^+(f+g) &= \sup\{L(h)\colon\ 0 \leqslant h \leqslant f + g\} \\ &= \sup\{L(h_1) + L(h_2)\colon\ 0 \leqslant h_1 \leqslant f,\ 0 \leqslant h_2 \leqslant g\} = L^+(f) + L^+(g). \end{aligned}$$

Now for all $f \in \mathcal{F}$ we set $L^+(f) = L^+(f^+) - L^+(f^-)$, where $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$. Note that if $f = f_1 - f_2$, where $f_1, f_2 \geqslant 0$, then we have $L^+(f) = L^+(f_1) - L^+(f_2)$. Indeed, obviously $f_1 + f^- = f_2 + f^+$ and hence $L^+(f_1) + L^+(f^-) = L^+(f_2) + L^+(f^+)$. It is clear that $L^+(tf) = tL^+(f)$ for all $t \in \mathbb{R}^1$ and $f \in \mathcal{F}$. The additivity of the functional L^+ follows from its additivity on nonnegative functions. Indeed, for any f and g we have $f = f^+ - f^-$ and $g = g^+ - g^-$, whence $f + g = (f^+ + g^+) - (f^- + g^-)$ and according to what has been said above we obtain

$$L^+(f+g) = L^+(f^+ + g^+) - L^+(f^- + g^-) = L^+(f) + L^+(g).$$

By definition, $L^+(f) \geqslant L(f)$ for nonnegative f, so the functional $L^- := L^+ - L$ is nonnegative. It is readily seen that L^- is given by the announced formula.

Finally, $\|L\| \leqslant \|L^+\| + \|L^-\| = L^+(1) + L^-(1)$. On the other hand,

$$L^+(1) + L^-(1) = 2L^+(1) - L(1) = \sup\{L(2\varphi - 1)\colon\ 0 \leqslant \varphi \leqslant 1\}$$
$$\leqslant \sup\{L(h):\ -1 \leqslant h \leqslant 1\} \leqslant \|L\|.$$

The theorem is proved for the lattice $\mathcal{F}$. The case of a general vector lattice is similar, see [**14**, p. 14]. □

Let us mention a remarkable result due to P. P. Korovkin on convergence of positive operators (for a proof, see [**339**, Chapter 1]).

6.10.62. Theorem. *Let T_n be positive linear operators on $C[0,1]$ in the sense that $T_n x \geqslant 0$ if $x \geqslant 0$ such that for the three functions $x_k(t) = t^k$, $k = 0, 1, 2$, we have $\|T_n x_k - x_k\| \to 0$ as $n \to \infty$. Then $\|T_n x - x\| \to 0$ for every $x \in C[0,1]$.*

The same assertion is true for the space $C_{2\pi}$ of continuous 2π-periodic functions if $x_1(t) = 1$, $x_2(t) = \sin t$ and $x_2(t) = \cos t$.

6.10(vi). Vector integration

Here we briefly discuss the Lebesgue integral for vector mappings, which is called in this case the Bochner integral (see [**147**], [**149**], [**164**], [**171**], [**397**], and [**614**]). Let $(\Omega, \mathcal{B})$ be a measurable space and let μ be a nonnegative measure on $\mathcal{B}$. First we suppose that the measure μ is bounded and then indicate the necessary changes for the general case.

Let E be a real or complex Banach space. Let $\mathcal{B}(E)$ denote the σ-algebra generated by open sets in E. This σ-algebra is called *Borel*.

6.10.63. Definition. *A mapping $f\colon \Omega \to E$ that is defined μ-almost everywhere is called μ-measurable if, for every $B \in \mathcal{B}(E)$, the set $f^{-1}(B)$ is μ-measurable.*

It is clear that if f is a μ-measurable mapping with values in a Banach space E, then the function $\omega \mapsto \|f(\omega)\|$ is also μ-measurable by the Borel measurability of open balls in E.

6.10.64. Proposition. *Let E be a separable Banach space and let $f\colon \Omega \to E$ be a mapping defined μ-almost everywhere. This mapping is μ-measurable precisely when for every $l \in E^*$ the scalar function $l \circ f$ is μ-measurable.*

A sufficient condition for the μ-measurability of f is the μ-measurability of all functions $l_n \circ f$, where $\{l_n\} \subset E^*$ is a countable set such that every element in E^* is the limit of a subsequence from $\{l_n\}$ in the weak-$*$ topology (such a set exists by the separability of E, see below).

PROOF. Let us consider the real case. The necessity of the indicated condition is clear from the fact that for every $l \in E^*$ and every open set $V \subset \mathbb{R}$ the set $f^{-1}\bigl(l^{-1}(V)\bigr)$ is μ-measurable, because $l^{-1}(V)$ is open. Let us show that the σ-algebra $\mathcal{B}_0$ generated by the halfspaces of the form $\{x\colon\, l(x) \leqslant r\}$ coincides with $\mathcal{B}(E)$. By the separability of the space E every open set in it equals the union of a countable collection of open balls with centers at points from $\{x_n\}$. Let U be such a ball. We show that $U \in \mathcal{B}_0$. Since U is the union of a sequence of closed balls with the same centers, we can pass to the closed ball U. It remains to represent it as the intersection of a countable collection of closed halfspaces. To this end, for every point $x_n \in E\backslash U$ and every closed ball $B(x_n, r_k)$ of rational radius r_k disjoint with U we find a halfspace $\Pi_{n,k}$ such that $U \subset \Pi_n$ and $B(x_n, r_k) \subset E\backslash\Pi_{n,k}$. We have $U = \bigcap_{n,k} \Pi_{n,k}$. Indeed, if $x \notin U$, then there exist x_n and r_k such that $x \in B(x_n, r_k)$ and $U \cap B(x_n, r_k) = \varnothing$, whence $x \notin \Pi_{n,k}$. Thus, $U \in \mathcal{B}_0$, which gives the equality $\mathcal{B}_0 = \mathcal{B}(E)$.

Let $\{l_n\} \subset E^*$ be a countable set such that every element l in E^* is the limit of a subsequence in $\{l_n\}$ in the weak-$*$ topology. The existence of such a set is obvious from the fact that E^* is the union of closed balls of radius n each of which is a metrizable compact space in the weak-$*$ topology (see Theorem 6.10.23). Then the function l is measurable with respect to the σ-algebra generated by $\{l_n\}$. Since the function l_n is measurable with respect to $\mathcal{B}_0$, so is the function l. □

Note that the measurability of f is equivalent to the measurability of the functions $g_n \circ f$, where $\{g_n\} \subset E^*$ is a countable set separating points in E, since for $\{l_n\}$ we can take the set of finite linear combinations of g_i with rational coefficients (the intersection of this set with the ball in E^* is dense in this ball in the weak-$*$ topology, which is easily verified).

For a broad class of measure spaces (in particular, for intervals with Borel measures) every measurable mapping with values in a Banach space automatically takes values in a separable subspace after redefinition on a set of measure zero (see Corollary 6.10.16 and Theorem 7.14.25 in [**73**]).

As in the case of scalar functions, μ-measurable vector valued mappings with finite sets of values will be called simple. For a such mapping ψ with values $y_1, \ldots, y_n$ on disjoint sets $\Omega_1, \ldots, \Omega_n$, the *Bochner integral* is defined by

$$\int_\Omega \psi(\omega)\, \mu(d\omega) := \sum_{i=1}^n y_i \mu(\Omega_i).$$

By the additivity of μ this integral is well-defined, i.e., does not depend on the partition of Ω into disjoint parts on which ψ is constant.

A sequence of simple mappings ψ_n is called *Cauchy or fundamental in mean* if for every $\varepsilon > 0$ there is N such that

$$\int_\Omega \|\psi_n(\omega) - \psi_k(\omega)\|\, \mu(d\omega) < \varepsilon \quad \text{if } n, k \geqslant N.$$

The sequence of integrals of ψ_n is Cauchy in E. Indeed, let $\Omega_1, \ldots, \Omega_N$ be disjoint measurable sets picked for fixed n and k such that ψ_n and ψ_k are constant on them. Such sets can be easily obtained by refining the sets on which ψ_n and

ψ_k are constant. Then we have the estimate which proves our claim:

$$\left\| \int_\Omega [\psi_n(\omega) - \psi_k(\omega)]\, \mu(d\omega) \right\| \leqslant \sum_{i=1}^{N} \|y_i^n - y_i^k\| \mu(\Omega_i) = \int_\Omega \|\psi_n(\omega) - \psi_k(\omega)\|\, \mu(d\omega).$$

6.10.65. Definition. *Let E be a Banach space. A mapping $f\colon \Omega \to E$ is called Bochner integrable if there exists a sequence of simple E-valued mappings ψ_n that converges to f μ-almost everywhere and is Cauchy in mean. The Bochner integral of f is defined as the limit of the integrals of ψ_n and denoted by the symbol $\int_\Omega f(\omega)\, \mu(d\omega)$.*

It follows from the scalar case that this definition is not ambiguous, since for every $l \in E^*$ the sequence of functions $l \circ \psi_n$ converges almost everywhere and is Cauchy in mean. Any integrable mapping f is measurable with respect to μ. Indeed, it follows from the definition that there exists a separable subspace $X_0 \subset X$ such that $f(\omega) \in X_0$ for μ-a.e. ω. Now the measurability of f follows from the proposition proved above and the measurability of the limit of a sequence of scalar measurable functions.

6.10.66. Example. Every bounded measurable mapping f with values in a separable Banach space E is Bochner integrable.

PROOF. Let us fix $n \in \mathbb{N}$. Let $\{x_i\}$ be a countable everywhere dense set in E. The space E is covered by the sequence of balls $B_{i,n} := B(x_i, 2^{-n})$. Let us find N_n such that

$$\mu\Bigl(\Omega \backslash \textstyle\bigcup_{i=1}^{N_n} f^{-1}(B_{i,n})\Bigr) < 2^{-n}.$$

Set $\Omega_{n,1} := f^{-1}(B_{1,n}), \Omega_{n,k} := f^{-1}(B_{k,n}) \backslash \Omega_{n,k-1}, k \leqslant N_n$. We define simple mappings ψ_n as follows. Let $\psi_n = x_k$ on $\Omega_{n,k}$ and let $\psi_n = 0$ outside $\Omega_n := \bigcup_{k=1}^{N_n} \Omega_{n,k}$. For every $\omega \in \Omega_{n,k}$ we have $\|\psi_n(\omega) - f(\omega)\| \leqslant 2^{-n}$. Hence this estimate is fulfilled on a set Ω_n with $\mu(\Omega \backslash \Omega_n) < 2^{-n}$. Therefore, $\lim\limits_{n\to\infty} \psi_n(\omega) = f(\omega)$ for μ-almost all ω. This is true for every ω from the set $\Omega' := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \Omega_n$ of full μ-measure. By hypothesis $\|f(\omega)\| \leqslant M$ for some M. Then $\|\psi_n(\omega)\| \leqslant M + 1$. For all $n > k$ and $\omega \in \Omega_n \cap \Omega_k$ we have $\|\psi_n(\omega) - \psi_k(\omega)\| \leqslant 2^{1-k}$, whence it follows that $\{\psi_n\}$ is Cauchy in mean. □

The estimate above implies that the function $\|f(\omega)\|$ is μ-integrable and

$$\left\| \int_\Omega f(\omega)\, \mu(d\omega) \right\| \leqslant \int_\Omega \|f(\omega)\|\, \mu(d\omega)$$

for any μ-integrable mapping f. The converse is also true.

6.10.67. Theorem. *Let f be a μ-measurable mapping with values in a separable Banach space E such that the function $\omega \mapsto \|f(\omega)\|$ is integrable with respect to μ. Then the mapping f is Bochner μ-integrable.*

PROOF. For every $n \in \mathbb{N}$ we find a measurable set Ω_n with $\mu(\Omega\backslash\Omega_n) < 2^{-n}$ and

$$\int_{\Omega\backslash\Omega_n} \|f(\omega)\| \, \mu(d\omega) < 2^{-n}.$$

Slightly decreasing the set Ω_n with the preservation of these bounds, we find simple mappings ψ_n for which $\|\psi_n(\omega) - f(\omega)\| \leqslant 2^{-n}$ for all $\omega \in \Omega_n$ and $\|\psi_n(\omega)\| \leqslant \|f(\omega)\| + 1$ for almost all ω. This is clear from the proof of the previous example. As above, we obtain convergence of the sequence $\{\psi_n\}$ to f almost everywhere and its fundamentality in mean. □

It follows from the definition that for every Bochner integrable mapping f with values in a Banach space E and every continuous linear operator T from E to a Banach space Y the mapping $T\circ f$ is also Bochner integrable and

$$\int_\Omega T\bigl(f(\omega)\bigr) \, \mu(d\omega) = T\biggl(\int_\Omega f(\omega) \, \mu(d\omega)\biggr).$$

Similarly to the scalar case the following assertion can be proved (Exercise 7.10.98).

6.10.68. Theorem. *The class $\mathcal{L}^1(\mu; E)$ of all Bochner μ-integrable everywhere defined mappings with values in a Banach space E is a linear space and the Bochner integral is linear on it. The set of μ-equivalence classes $L^1(\mu; E)$ with the norm*

$$\|f\|_{L^1(\mu;E)} := \int_\Omega \|f(\omega)\| \, \mu(d\omega),$$

given by means of a representative of the equivalence class is a Banach space. In addition, for every $p \in [1, +\infty)$ the subspace $L^p(\mu; E)$ in $L^1(\mu; E)$ corresponding to mappings with a finite norm

$$\|f\|_{L^p(\mu;E)} := \biggl(\int_\Omega \|f(\omega)\|^p \, \mu(d\omega)\biggr)^{1/p}$$

is also a Banach space.

Similarly to the scalar case one can include in $\mathcal{L}^1(\mu; E)$ mappings that are not defined on a measure zero set and coincide outside a measure zero set with a mapping from the previously defined class. For the same reasons as for real functions, this set will not be a linear space, although one can take sums of such mappings and multiply them by constants (on a set of full measure). This is a matter of convenience, but the difference disappears after passage to the factor-space $L^1(\mu; E)$.

In case of an infinite measure the construction of the Bochner integral is similar, but in the definition of a simple mapping it is required in addition that it must be zero outside a set of finite measure.

In applications it is often useful to work with weaker notions of the scalar integrability or the Pettis integrability.

A mapping f with values in a Banach space E is called *scalarly integrable* if for every $l \in E^*$ the function $l\circ f$ is integrable with respect to μ and there is an element $h \in E$, called the *scalar integral* of f, such that the integral of the function $l \circ f$ equals $l(h)$. If f is scalarly integrable on every measurable set, then it is called *Pettis integrable* and its scalar integral is called the *Pettis integral* of f.

If E is separable, then the scalar integrability implies the measurability of f, but in the general case it does not yield the Bochner integrability. For example, let the measure μ on l^2 be concentrated at the points of the form ne_n and $-ne_n$, where $\{e_n\}$ is the standard basis in l^2, and let the measure of such point be n^{-2}. The mapping $f(x) = x$ has zero Pettis integral, since for every $y = (y_n) \in l^2$ the series with the general term $y_n n^{-1}$ is absolutely convergent. However, the function $\|x\|$ is not integrable with respect to μ due to divergence of the series with the general term n^{-1}.

6.10(vii). The Daniell integral

The construction of the Lebesgue integral presented in this book is based on a preliminary study of measure. It is possible, however, to act in the opposite order: to define measure by means of integral. At the basis of this approach there is the following result due to Daniell. Its formulation employs the notion of a vector lattice of functions, i.e., a linear space $\mathcal{F}$ of real functions on a nonempty set Ω such that $\max(f, g) \in \mathcal{F}$ for all $f, g \in \mathcal{F}$ (this is equivalent to the closedness of $\mathcal{F}$ with respect to taking absolute values). Vector lattices of functions considered here are a particular case of abstract vector lattices mentioned in §5.6(iii). Suppose that on $\mathcal{F}$ we are given a nonnegative linear functional L, i.e., $L(f) \geqslant 0$ for all $f \geqslant 0$, and that $L(f_n) \to 0$ for every sequence $\{f_n\} \subset \mathcal{F}$ pointwise decreasing to zero. Such a functional is called the *Daniell integral*. Our nearest goal is to extend L to a larger domain of definition $\mathcal{L}$ such that the extension will possess the main properties of the integral, i.e., admit analogs of monotone and dominated convergence theorems and $\mathcal{L}$ will be complete. An example which can be taken as model is an extension of the Riemann integral from the set of step or continuous functions. Then we clarify the connections between the Daniell integral and the true integral with respect to a measure.

6.10.69. Definition. *A set $S \subset \Omega$ will be called L-zero if there exists a nondecreasing sequence of nonnegative functions $f_n \in \mathcal{F}$ for which $\sup_n L(f_n) < \infty$ and $\sup_n f_n(x) = +\infty$ on S.*

This definition is inspired by the monotone convergence theorem.

6.10.70. Lemma. (i) *The union of countably many L-zero sets S_k is an L-zero set.*

(ii) *A set S is L-zero precisely when for every $\varepsilon > 0$ there exists a nondecreasing sequence of nonnegative functions $f_n \in \mathcal{F}$ with $L(f_n) < \varepsilon$ and $\sup_n f_n(x) \geqslant 1$ on S.*

(iii) *If $f \in \mathcal{F}$ and $f \geqslant 0$ outside an L-zero set, then $L(f) \geqslant 0$. If $f, g \in \mathcal{F}$ and $f \leqslant g$ outside an L-zero set, then $L(f) \leqslant L(g)$.*

PROOF. (i) For any fixed k we take functions $f_{k,n} \in \mathcal{F}$, $n \in \mathbb{N}$, such that $L(f_{k,n}) \leqslant 2^{-k}$ and $\sup_n f_{k,n}(x) = \infty$ on S_k. Let $g_n = f_{1,n} + \cdots + f_{n,n}$. Then $g_n \in \mathcal{F}$, $L(g_n) \leqslant 1$ and $\sup_n g_n(x) = +\infty$ on $\bigcup_{k=1}^{\infty} S_k$, since the latter is true on every S_k.

(ii) If S is L-zero and $\{f_n\} \subset \mathcal{F}$ is an increasing sequence with the properties indicated in the definition and $L(f_n) \leqslant M$, then, for any given $\varepsilon > 0$, we take the function $\varepsilon M^{-1} f_n$. Conversely, if the condition in (i) is fulfilled, then for every k there exists an increasing sequence $\{f_{k,n}\} \subset \mathcal{F}$ such that $L(f_{k,n}) \leqslant 2^{-k}$ for all n and $\sup_n f_{k,n} \geqslant k$ on S. Let $g_n = f_{1,1} + \cdots + f_{n,n}$. Then $g_n \in \mathcal{F}$, $0 \leqslant g_n \leqslant g_{n+1}$, $L(g_n) \leqslant 1$, and $\sup_n g_n = +\infty$ on S. Hence S is L-zero.

(iii) Let $S = \{f < 0\}$ and $c = L(f) < 0$. Let us take an increasing sequence of nonnegative functions $f_n \in \mathcal{F}$ with $L(f_n) \leqslant |c|/2$ and $\sup_n f_n(x) = \infty$ for all $x \in S$. Then the sequence of functions $f + f_n$ is increasing and its finite or infinite limit is everywhere nonnegative, since the function f is finite on S and nonnegative outside S. In addition, $L(f + f_n) \leqslant -|c|/2$. Set $\varphi_n = (f + f_n)^-$. Then $\varphi_n \in \mathcal{F}$ and the functions φ_n are pointwise increasing to zero. Hence $L(\varphi_n) \to 0$ contrary to that $L(\varphi_n) \leqslant L(f + f_n) \leqslant -|c|/2$. Thus, $c = 0$. □

This yields the following stronger continuity property of L.

6.10.71. Corollary. *Let $\{f_n\} \subset \mathcal{F}$ and $f_n \downarrow 0$ outside some L-zero set S. Then $L(f_n) \downarrow 0$.*

PROOF. Set $g_n = \min(f_1, \ldots, f_n)$. Then $g_n \in \mathcal{F}$. Outside S we have $g_n = f_n$. By the lemma $L(f_n) = L(g_n)$. In addition, $\{g_n\}$ is everywhere decreasing. If $L(g_n) \geqslant c > 0$ for all n, then we take an increasing sequence of nonnegative functions $\varphi_n \in \mathcal{F}$ with $L(\varphi_n) < c/2$ and $\sup_n \varphi_n(x) = \infty$ for all $x \in S$. Then the functions $g_n - \varphi_n \in \mathcal{F}$ decrease everywhere to a nonpositive limit, because outside S the functions g_n decrease to zero and on S we have $g_n - \varphi_n \leqslant g_1 - \varphi_n \to -\infty$. The functions $(g_n - \varphi_n)^+ \in \mathcal{F}$ pointwise decrease to zero, whence we obtain $L\bigl((g_n - \varphi_n)^+\bigr) \to 0$. This contradicts the estimate $L\bigl((g_n - \varphi_n)^+\bigr) \geqslant L(g_n - \varphi_n) \geqslant c/2$. □

Denote by $\mathcal{L}^{\uparrow}$ the class of all functions $f\colon \Omega \to (-\infty, +\infty]$ for which there exists a sequence of nonnegative functions $f_n \in \mathcal{F}$ such that outside some L-zero set we have $f(x) = \lim_{n\to\infty} f_n(x)$ and $f_n(x) \leqslant f_{n+1}(x)$ for all n and the sequence $\{L(f_n)\}$ is bounded. Set

$$L(f) := \lim_{n\to\infty} L(f_n).$$

The limit exists, since by assertion (iii) of the lemma above the sequence $\{L(f_n)\}$ increases. The next lemma shows that L is well-defined on $\mathcal{L}^{\uparrow}$.

6.10.72. Lemma. (i) *Suppose that $\{f_n\}$ and $\{g_n\}$ are two sequences from $\mathcal{F}$ such that outside some L-zero set S they increase and satisfy the condition*

$$\lim_{n\to\infty} f_n(x) \leqslant \lim_{n\to\infty} g_n(x),$$

where also infinite limits are allowed. Then we have $\lim\limits_{n\to\infty} L(f_n) \leqslant \lim\limits_{n\to\infty} L(g_n)$. *In particular, these limits coincide if outside* S *the sequences* $\{f_n\}$ *and* $\{g_n\}$ *increase to a common limit.*

(ii) *Every function from* $\mathcal{L}^{\uparrow}$ *is finite outside some* L*-zero set.*

PROOF. (i) For any fixed n the functions $f_n - g_k$ outside S decrease to a nonpositive limit. Hence $(f_n - g_k)^+ \downarrow 0$ outside S, which by the corollary above gives $L\bigl((f_n - g_k)^+\bigr) \downarrow 0$. Hence $\lim\limits_{k\to\infty} L(f_n - g_k) \leqslant 0$, i.e., $L(f_n) \leqslant \lim\limits_{k\to\infty} L(g_k)$, which gives the desired inequality.

(ii) Suppose that functions $f_n \in \mathcal{F}$ increase to f outside an L-zero set S and $L(f_n) \leqslant C$. Set $g_n = \max(f_1, \ldots, f_n)$. Then $g_n \in \mathcal{F}$, $g_n \leqslant g_{n+1}$, and $g_n = f_n$ outside S. The set $Z = \{x\colon\ \sup_n g_n(x) = \infty\}$ is L-zero by definition. The function f is finite outside the set $S \cup Z$. □

It follows from the established facts that if $f \in \mathcal{L}^{\uparrow}$ and a function $g \geqslant 0$ equals f on the complement of an L-zero set, then $g \in \mathcal{L}^{\uparrow}$ and $L(g) = L(f)$. Hence every function from $\mathcal{L}^{\uparrow}$ can be made everywhere finite without changing $L(f)$.

Let $\mathcal{L}$ denote the set of all real functions f representable as the difference $f = f_1 - f_2$ of two *everywhere finite* functions $f_1, f_2 \in \mathcal{L}^{\uparrow}$. For such functions we set $L(f) := L(f_1) - L(f_2)$. This value is well-defined, which is verified in the next theorem.

On $\mathcal{L}$ we can introduce an equivalence relation by declaring to be equivalent those functions which coincide outside an L-zero set. Then the set $\widetilde{\mathcal{L}}$ of equivalence classes becomes a metric space with the metric $d_L(f,g) := L(|f-g|)$. In addition, $\widetilde{\mathcal{L}}$ is a linear space and L is naturally defined on $\widetilde{\mathcal{L}}$. It is clear that $\mathcal{F}$ is everywhere dense in $\widetilde{\mathcal{L}}$.

6.10.73. Theorem. (i) *The functional* L *on* $\mathcal{L}$ (*or* $\widetilde{\mathcal{L}}$) *is well-defined and linear.*

(ii) *If* $f \in \mathcal{L}$, *then* $|f| \in \mathcal{L}$ *and* $|L(f)| \leqslant L(|f|)$.

(iii) *The assertions of the monotone convergence, Lebesgue dominated convergence and Fatou theorems hold for* L *with* L *in place of the integral*: *if* $f_n \in \mathcal{L}$, $f_n \to f$ *outside an* L*-zero set and either there is a function* $\Phi \in \mathcal{L}$ *such that* $|f_n| \leqslant \Phi$ *outside an* L*-zero set or* $\{f_n\}$ *is increasing outside an* L*-zero set and* $\sup_n L(f_n) < \infty$, *then* $f \in \mathcal{L}$ *and* $\lim\limits_{n\to\infty} L(f_n) = L(f)$; *if* $f_n \geqslant 0$ *and* $\sup_n L(f_n) < \infty$, *then* $f \in \mathcal{L}$ *and* $L(f_n) \leqslant \liminf_n L(f_n)$.

In addition, the space $\widetilde{\mathcal{L}}$ *is complete with respect to the metric* d_L.

PROOF. (i) It is readily seen that $f + g \in \mathcal{L}^{\uparrow}$ for all $f, g \in \mathcal{L}^{\uparrow}$, moreover, $L(f+g) = L(f) + L(g)$. If $f_1, f_2, g_1, g_2 \in \mathcal{L}^{\uparrow}$ and $f_1 - f_2 = g_1 - g_2$, then $L(f_1) + L(g_2) = L(g_1) + L(f_2)$, which shows that L is well-defined on $\mathcal{L}$. Since $L(\alpha f) = \alpha L(g)$ for all $f \in \mathcal{L}^{\uparrow}$ and all $\alpha \geqslant 0$, then this is true for all $f \in \mathcal{L}$ and $\alpha \in \mathbb{R}$. If $f, g \in \mathcal{L}$ and $f = f_1 - f_2$, $g = g_1 - g_2$, where $f_1, f_2, g_1, g_2 \in \mathcal{L}^{\uparrow}$, then by the facts already proved above we have $f + g = f_1 + g_1 - (f_2 + g_2) \in \mathcal{L}$ and $L(f+g) = L(f_1 + g_1) - L(f_2 + g_2) = L(f) + L(g)$.

The closedness of $\mathcal{F}$ with respect to the operations $(\varphi,\psi)\mapsto\min(\varphi,\psi)$ and $(\varphi,\psi)\mapsto\max(\varphi,\psi)$ yields the closedness of $\mathcal{L}^{\uparrow}$ with respect to these operations. Hence $\min(f_1,f_2)\in\mathcal{L}^{\uparrow}$ and $\max(f_1,f_2)\in\mathcal{L}^{\uparrow}$, whence

$$|f|=|f_1-f_2|=\max(f_1,f_2)-\min(f_1,f_2)\in\mathcal{L}.$$

Finally, for the proof of the estimate $|L(f)|\leqslant L(|f|)$ it suffices to verify that $L\varphi\geqslant 0$ if $\varphi\in\mathcal{L}$ and $\varphi\geqslant 0$. Then $\varphi=\psi_1-\psi_2$, where $\psi_1,\psi_2\in\mathcal{L}^{\uparrow}$ and $\psi_2\leqslant\psi_1$. Hence we can apply assertion (i) of the lemma above.

(ii) Let $\{f_n\}\subset\mathcal{L}$ be increasing outside an L-zero set and let the sequence $\{L(f_n)\}$ be bounded. For every n we find functions $f_{n,k}\in\mathcal{F}$ increasing to f_n outside some L-zero set S_n. Let $g_n=\max_{k,m\leqslant n}f_{m,k}$. Then $g_n\in\mathcal{F}$, the sequence $\{g_n\}$ is increasing and $\{L(g_n)\}$ is bounded. Hence $f=\lim\limits_{n\to\infty}g_n\in\mathcal{L}^{+}$ and $L(f)=\lim\limits_{n\to\infty}L(g_n)$. It is clear that $f_n(x)\to f(x)$ outside an L-zero set and $L(f)=\lim\limits_{n\to\infty}L(f_n)$, since $L(g_n)\leqslant L(f_n)$ and $L(f_n)=\lim\limits_{k\to\infty}L(f_{n,k})$. Fatou's theorem is deduced precisely as in the case of the Lebesgue integral.

Let $f_n(x)\to f(x)$ and $|f_n(x)|\leqslant\Phi(x)$ outside an L-zero set, where $f_n,\Phi\in\mathcal{L}$. Set $\varphi_n(x):=\inf_{k\geqslant n}f_k(x)$, $\psi_n(x):=\sup_{k\geqslant n}f_k(x)$. Then outside an L-zero set we have $\varphi_n\leqslant f_n\leqslant\psi_n$, $\varphi_n\geqslant-\Phi$, $\psi_n\leqslant\Phi$, $\varphi_n\uparrow f$, $\psi_n\downarrow f$. Hence $f\in\mathcal{L}$ and $L(f)=\lim\limits_{n\to\infty}L(\varphi_n)=\lim\limits_{n\to\infty}L(\psi_n)$, which gives $L(f)=\lim\limits_{n\to\infty}L(f_n)$.

Let now a sequence $\{f_n\}\subset\mathcal{L}$ be Cauchy in the metric d_L. Passing to a subsequence, we can assume that $d_L(f_n,f_{n+1})\leqslant 2^{-n}$. As shown above, the series of $|f_n-f_{n-1}|$, where $f_0:=0$, converges outside some L-zero set S to an element $\Phi\in\mathcal{L}$. Then the sums $f_n=\sum_{k=1}^{n}(f_k-f_{k-1})$ converge to a finite limit f outside S. Since $|f_n|\leqslant\Phi$, we conclude that $\{f_n\}$ converges to f in $\widetilde{\mathcal{L}}$. □

The Daniell integral possesses the most important properties of the Lebesgue integral, so the question arises whether it can be defined as the integral with respect to some countably additive measure. Moreover, some measure arises automatically. Indeed, denote by $\mathcal{R}_L$ the class of all sets $E\in\Omega$ for which $I_E\in\mathcal{L}$ and let $\nu(E):=L(I_E)$. It follows from the previous theorem that $\mathcal{R}_L$ is a δ-ring and that the measure ν on it is countably additive. However, in the general case the integral with respect to this measure can fail to coincide with L. As the following example shows, without additional assumptions it is not always possible to represent L as the Lebesgue integral with respect to a countably additive measure.

6.10.74. Example. Let $\mathcal{F}$ be the set of all finite real functions f on $[0,1]$ with the following property: for some number $\alpha=\alpha(f)$, the set $\{t\colon\ f(t)\neq\alpha(1+t)\}$ is a first category set. Let $L(f):=\alpha$. Then $\mathcal{F}$ is a vector lattice of functions with the natural order relation from $\mathbb{R}^{[0,1]}$, L is a nonnegative linear functional on $\mathcal{F}$ and $L(f_n)\to 0$ for every sequence of functions $f_n\in\mathcal{F}$ pointwise decreasing to zero, but even on the subspace of all bounded functions in $\mathcal{F}$ the functional L cannot be defined as the integral with respect to a countably additive measure.

PROOF. We observe that for every function $f\in\mathcal{F}$ there is only one number α with the indicated property, since the interval is not a first category set. Hence the function L is well-defined. For any $f\in\mathcal{F}$ let $E_f:=\{t\colon\ f(t)\neq\alpha(1+t)\}$

for the number α corresponding to f. If $f, g \in \mathcal{F}$ and $\alpha = \alpha(f)$, $\beta = \alpha(g)$ are the corresponding numbers, then the set $E_f \cup E_g$ has the first category and outside this set we have $f(t) + g(t) = (\alpha + \beta)(1 + t)$. For every scalar c we have $cf(t) = c\alpha(1 + t)$ outside the set E_f. Thus, $\mathcal{F}$ is a linear space. It is readily seen that $|f| \in \mathcal{F}$ for all $f \in \mathcal{F}$. It is clear from what has been said that the function L is linear. For all $f \geqslant 0$ we have $L(f) \geqslant 0$. If functions $f_n \in \mathcal{F}$ pointwise decrease to zero, then the union of the sets E_{f_n} is a first category set. Hence there exists a point t such that $L(f_n) = f_n(t)/(1+t)$ for all n at once, whence $\lim\limits_{n\to\infty} L(f_n) = 0$.

Suppose now that there exists a measure μ on $\sigma(\mathcal{F})$ with values in $[0, +\infty]$ such that every bounded function f from $\mathcal{F}$ belongs to $\mathcal{L}^1(\mu)$ and $L(f)$ coincides with the integral of f with respect to the measure μ. The function $\psi\colon\ t \mapsto 1 + t$ belongs to $\mathcal{F}$, whence it follows that all open sets from $[0, 1]$ belong to $\sigma(\mathcal{F})$. By the estimate $\psi \geqslant 1$ we obtain $\mu([0, 1]) \leqslant L(\psi) = 1$. Thus, the restriction of μ to $\mathcal{B}([0, 1])$ is a finite measure. Hence there exists a Borel first category set E such that $\mu([0, 1]\backslash E) = 0$. Indeed, we can take the union of nowhere dense compact sets K_n with $\mu([0, 1]\backslash K_n) < 1/n$, which are constructed by means of deleting sufficiently small intervals with centers at the points of an everywhere dense countable set of μ-measure zero. Let us consider the following function f: $f(t) = 0$ if $t \in E$, $f(t) = 1 + t$ if $t \notin E$. It is clear that $f \in \mathcal{F}$ and $L(f) = 1$. On the other hand, the integral of f with respect to the measure μ is zero, which gives a contradiction. □

In this example the measure ν generated by L on the δ-ring $\mathcal{R}_L$ is identically zero. Indeed, here L-zero sets are first category sets, since if $\alpha(f_n) \leqslant 1/3$, then $f_n(t) \leqslant 2/3$ outside a first category set. Hence the class $\mathcal{L}$ coincides with $\mathcal{F}$. The indicator function of a set can belong to $\mathcal{F}$ only for $\alpha = 0$, i.e., $\mathcal{R}_L$ consists of first category sets and they are zero sets.

One should bear in mind that the measure ν can be zero also in the case where L is given as the integral with respect to some nonzero measure μ. For example, let us take for $\mathcal{F}$ the one-dimensional linear space generated by the function $f(t) = t$ on $(0, 1)$ and for L take the Riemann integral, i.e., $L(\alpha f) = \alpha/2$. Then $\mathcal{R}_L$ consists of the empty set and $\nu = 0$.

We now prove that adding one simply stated condition, fulfilled in all applications, leads to the effect that the Daniell integral is given by the Lebesgue integral with respect to some measure. This is the so-called *Stone condition*:

$$\min(f, 1) \in \mathcal{F} \quad \text{for all } f \in \mathcal{F}.$$

This condition is trivially fulfilled if the lattice of functions $\mathcal{F}$ contains 1. A nontrivial example of a lattice with the Stone condition, but without 1, is the space of continuous functions with compact support on $\mathbb{R}^n$. The space $\mathcal{F}$ from the previous example and its subspace consisting of bounded functions obviously do not satisfy the Stone condition.

6.10.75. Theorem. *Let $\mathcal{F}$ satisfy the Stone condition and let L be a nonnegative linear functional on $\mathcal{F}$ such that $L(f_n) \to 0$ for every sequence of functions $f_n \in \mathcal{F}$ pointwise decreasing to zero. Then there exists a countably additive* *measure* μ *on* $\sigma(\mathcal{F})$ *with values in* $[0,+\infty]$ *such that* $\mathcal{F}\subset\mathcal{L}^1(\mu)$ *and*

$$L(f)=\int_\Omega f(\omega)\,\mu(d\omega),\quad f\in\mathcal{F}. \tag{6.10.9}$$

In addition, $\widetilde{\mathcal{L}}=L^1(\mu)$ *and such a measure* μ *is unique on the* σ*-ring generated by the sets of the form* $\{f>c\}$*, where* $f\in\mathcal{F}$ *and* $c>0$*. Finally, if* $1\in\mathcal{F}$*, then the measure* μ *is finite.*

PROOF. The measure ν constructed above on the δ-ring $\mathcal{R}_L$ uniquely extends to a countably additive measure μ (with values in $[0,+\infty]$) on the σ-ring $\mathcal{R}_L^\sigma$ generated by $\mathcal{R}_L$. In addition, it extends (but not necessarily uniquely) to a countably additive measure μ on the σ-algebra generated by $\mathcal{R}_L$. Let us show that the latter coincides with the σ-algebra $\sigma(\mathcal{L})$ generated by $\mathcal{L}$. To this end we observe that one can easily derive from the Stone condition that $\min(f,1)\in\mathcal{L}$ for all $f\in\mathcal{L}$. This gives the inclusion $E_c:=f^{-1}(c,+\infty)\in\mathcal{R}_L$ for all $c>0$ and $f\in\mathcal{L}$. Indeed, it suffices to verify this for $c=1$. Then the functions $\varphi_n:=\min\bigl(1,nf-n\min(1,f)\bigr)\in\mathcal{L}$ increase pointwise to I_{E_1}, $0\leqslant\varphi_n\leqslant 1$ and $\varphi_n\leqslant|f|$, whence $E(\varphi_n)\leqslant L(|f|)$. Therefore, $f^{-1}\bigl((\alpha,\beta]\bigr)\in\mathcal{R}_L$ whenever $0<\alpha<\beta$ and $f\in\mathcal{L}$. It follows from this that all functions from $\mathcal{L}$ are measurable with respect to the σ-algebra generated by $\mathcal{R}_L$. Let $f\in\mathcal{L}$ and $0\leqslant f\leqslant 1$. Set $f_k:=\sum_{n=1}^{2^k-1}n2^{-k}I_{(n2^{-k},(n+1)2^{-k}]}$. Then $f_k\in\mathcal{L}$, $f_k\to f$ and $f_k\leqslant f$. Hence

$$\int_\Omega f_k\,d\mu=L(f_k)\to L(f).$$

Therefore, the function f is integrable with respect to μ and its integral is $L(f)$. Now for every nonnegative function $f\in\mathcal{L}$ we obtain $\min(f,n)\in L^1(\mu)$ and the integral of $\min(f,n)$ equals $L\bigl(\min(f,n)\bigr)$, which gives the μ-integrability of f and the equality of its integral to $L(f)$ by the monotone convergence theorem for the integral and for L. Then this equality remains true for all $f\in\mathcal{L}$. Clearly, $\sigma(\mathcal{F})\subset\sigma(\mathcal{L})$ and $\mathcal{F}\subset\mathcal{L}$. The uniqueness of a representing measure on the σ-ring generated by the sets of the form $\{f>c\}$, where $f\in\mathcal{F}$ and $c>0$, is clear from the proof. Finally, if $1\in\mathcal{F}$, then $\mu(\Omega)=L(1)<\infty$. □

On $\sigma(\mathcal{F})$ the measure μ is not always unique. For example, if $\mathcal{F}$ is the space of all finite Lebesgue integrable functions f on $[0,+\infty)$ with $f(0)=0$ and L is the Lebesgue integral, then for μ we take any measure $\lambda+c\delta_0$, where λ is Lebesgue measure, δ_0 is Dirac's measure at zero and $c\geqslant 0$.

Typical applications of the Daniell–Stone method are connected with extensions of functionals on spaces of continuous functions.

6.10.76. Theorem. (THE RIESZ THEOREM) *Let* K *be a compact space and let* L *be a linear functional on* $C(K)$ *such that* $L(f)\geqslant 0$ *if* $f\geqslant 0$*. Then on the* σ*-algebra generated by all continuous functions on* K *there exists a nonnegative finite measure* μ *such that*

$$L(f)=\int_K f(x)\,\mu(dx),\quad f\in C(K).$$

PROOF. By assumption we have $|L(f)| \leqslant L(1) \max_x |f(x)|$. It remains to observe that if functions $f_n \in C(K)$ decrease pointwise to zero, then by Dini's theorem $\max_x |f_n(x)| \to 0$ and hence $L(f_n) \to 0$. □

Note that the measure μ uniquely extends to a measure on the Borel σ-algebra of $\mathcal{B}(K)$ (in case of nonmetrizable K the latter can be larger than the σ-algebra $\mathcal{B}a(K)$ generated by $C(K)$) with the following regularity property: $\mu(B) = \sup\{\mu(C)\colon C \subset B \text{ is compact}\}$ for all Borel sets $B \subset K$. A proof can be found in [**73**, Chapter 7]. If K is metrizable, then $\mathcal{B}(K) = \mathcal{B}a(K)$.

A more general example (with a possibly infinite measure) is obtained if we consider a positive functional on the space $C_0(T)$ of all continuous functions with compact support on a locally compact space T. For example, in this way we can extend the Riemann integral to the Lebesgue integral on $\mathbb{R}^n$ or on a manifold. By the way, it is clear from this why in the definition of the integral on $\mathcal{L}^{\uparrow}$ we used L-zero sets: if we take only pointwise limits of continuous functions, then not every Lebesgue integrable function will be equal almost everywhere to a function from the obtained class. Finally, we observe that if $1 \in \mathcal{F}$, then representation (6.10.9) takes place without the assumption that L is nonnegative. For this the functional L satisfying the condition $L(f_n) \to 0$ as $f_n \downarrow 0$ can be decomposed into the difference of two nonnegative functionals satisfying the same condition (see Theorem 6.10.61 and [**73**, §7.8]).

6.10(viii). Interpolation theorems

Here we prove the M. Riesz and Thorin interpolation theorem, which is one of the most important results in the theory of interpolation of linear operators. Let μ and ν be nonnegative measures on measurable spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$.

6.10.77. Theorem. *Let $p_0, q_0, p_1, q_1 \in [1, +\infty]$, where $p_0 \neq p_1$ and $q_0 \neq q_1$. Suppose we are given a linear mapping*

$$T\colon\ L^{p_0}(\mu) \cap L^{p_1}(\mu) \to L^{q_0}(\nu) \cap L^{q_1}(\nu),$$

where we consider complex spaces, such that

$$\|Tf\|_{L^{q_0}(\nu)} \leqslant M_0 \|f\|_{L^{p_0}(\mu)} \quad \text{and} \quad \|Tf\|_{L^{q_1}(\nu)} \leqslant M_1 \|f\|_{L^{p_1}(\mu)}.$$

Then T extends to an operator between the spaces $L^p(\mu)$ and $L^q(\nu)$ with the norm $M \leqslant M_0^{1-\theta} M_1^{\theta}$ provided that $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

PROOF. It is clear that T extends to an operator from $L^{p_0}(\mu)$ to $L^{q_0}(\nu)$ and from $L^{p_1}(\mu)$ to $L^{q_1}(\nu)$ with norms not exceeding M_0 and M_1. The point (p^{-1}, q^{-1}) belongs to the interval joining the points (p_0^{-1}, q_0^{-1}) and (p_1^{-1}, q_1^{-1}) in the plane. This explains the term "interpolation". If the measure μ is finite and $p_0 < p_1$, then p^{-1} is between p_1^{-1} and p_0^{-1} and hence $L^p(\mu)$ is between $L^{p_1}(\mu)$ and $L^{p_0}(\mu)$, i.e., T can be restricted from $L^{p_0}(\mu)$ to $L^p(\mu)$. However, it is not obvious at all that $L^p(\mu)$ will take values in $L^q(\nu)$. We observe that $1 < p < \infty$

and $1 < q < \infty$. Hence out further considerations can be conducted for simple integrable functions. Whenever $0 \leqslant \operatorname{Re} z \leqslant 1$, let

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}, \quad \frac{1}{q'(z)} = \frac{1-z}{q'_0} + \frac{z}{q'_1},$$

$$\varphi(z) = |f|^{p/p(z)} f/|f|, \quad \psi(z) = |g|^{q'/q'(z)} g/|g|,$$

where f and g are simple integrable functions on Ω_1 and Ω_2, respectively, and $\|f\|_{L^p(\mu)} = \|g\|_{L^{q'}(\nu)} = 1$. Then the function

$$F(z) = \int_{\Omega_2} T\varphi(z)\psi(z)\, d\nu$$

is analytic in the open strip $0 < \operatorname{Re} z < 1$ and continuous and bounded in its closure. Straightforward calculations show that

$$\|\varphi(it)\|_{L^{p_0}(\mu)} = \|\varphi(1+it)\|_{L^{p_1}(\mu)} = \|\psi(it)\|_{L^{q'_0}(\nu)} = \|\psi(1+it)\|_{L^{q'_1}(\nu)} = 1.$$

By assumption, $|F(it)| \leqslant M_0$ and $|F(1+it)| \leqslant M_1$. Note that $F(\theta)$ is the integral of $(Tf)g$ with respect to the measure ν, since $\varphi(\theta) = f$, $\psi(\theta) = g$. Since the norm of T as an operator from $L^p(\mu)$ to $L^q(\nu)$ is the supremum of the values $|F(\theta)|$ over f and g of the indicated form, for obtaining the desired estimate it suffices to apply the Hadamard three lines theorem from complex analysis, which gives the estimate $|F(\theta + it)| \leqslant M_0^{1-\theta} M_1^{\theta}$ for all $t \in \mathbb{R}$. □

In the real case the same is true with the estimate $M \leqslant 2M_0^{1-\theta} M_1^{\theta}$.

6.10.78. Example. If an operator T belongs to the spaces $\mathcal{L}\big(L^p(\mu), L^p(\nu)\big)$ and $\mathcal{L}\big(L^q(\mu), L^q(\nu)\big)$ with $p < q$, then its norms are finite in $\mathcal{L}\big(L^r(\mu), L^r(\nu)\big)$ with $r \in [p, q]$. If T is bounded as a mapping between $L^1(\mu)$ and $L^1(\nu)$ and also between $L^\infty(\mu)$ and $L^\infty(\nu)$, then T has a finite norm in $\mathcal{L}\big(L^p(\mu), L^p(\nu)\big)$ for all $p \in (1, \infty)$. This norm does not exceed 1 if this is true for the norms on L^1 and on L^∞ (over $\mathbb{C}$).

For other results, including the Marcinkiewicz interpolation theorem, see Bennett, Sharpley [**54**], Bergh, Löfström [**61**], Krein, Petunin, Semenov [**353**], Lunardi [**398**], and Triebel [**607**].

Exercises

6.10.79.° Give an example of a discontinuous function $f\colon \mathbb{R} \to \mathbb{R}$ with a closed graph.

HINT: consider the function $f(x) = 1/x$, $x \neq 0$, $f(0) = 0$.

6.10.80.° Let X, Y be normed spaces and let $\lim\limits_{n\to\infty} A_n x = Ax$ for all $x \in X$, where $A, A_n \in \mathcal{L}(X, Y)$. Prove that $\|A\| \leqslant \liminf_{n\to\infty} \|A_n\|$, and give an example where the equality fails.

HINT: consider the one-dimensional projections generated by the standard orthonormal basis in l^2.

6.10.81.° Let X be a normed space and $f \in X^*$, $\|f\| = 1$. Prove that for every x the equality $\operatorname{dist}(x, \operatorname{Ker} f) = |f(x)|$ holds.

6.10.82. Let Y be a closed subspace in a Banach space X and $x_0 \in X\backslash Y$. Prove that there exists a functional $l \in X^*$ such that $\|l\| = 1$, $l|_Y = 0$ and $l(x_0) = \text{dist}\,(x_0, Y)$.

HINT: let $d = \text{dist}\,(x_0, Y)$; on the linear span of Y and x_0 consider the functional $l(y + tx_0) = dt$. Clearly, $l|_Y = 0$. Since $\|y + tx_0\| \geqslant d|t|$, one has $\|l\| \leqslant 1$. Take $\{y_n\} \subset Y$ with $\|y_n - x_0\| \to d$, which gives $d = |l(x_0) - l(y_n)| \leqslant \|l\|\,\|x_0 - y_n\| \to d\|l\|$, whence $\|l\| = 1$. Now extend the functional l to X with the same norm.

6.10.83. Show that the cardinality of an infinite-dimensional Banach space is always strictly smaller than that of its algebraic dual.

HINT: use a Hamel basis and Exercise 5.6.55 along with Cantor's theorem.

6.10.84. Suppose that a linear space E is equipped with two non-equivalent norms. Prove that the duals to E with respect to these norms are different as sets of linear functions.

HINT: if the unit ball U with respect to the first norm p_1 is not bounded with respect to the second norm p_2, then it is not weakly bounded in (E, p_2) and hence there exists a functional l continuous in the norm p_2 such that $\sup_{u \in U} |l(u)| = \infty$.

6.10.85. Let Y be a linear subspace in a normed space X with separable dual X^*. Prove that Y^* is separable.

HINT: apply the Hahn–Banach theorem.

6.10.86. Let X be a normed space such that X^* is norm separable. Prove that X is separable.

HINT: take a countable set $\{l_n\}$ dense in the unit sphere of X^*; for every n find x_n with $\|x_n\| = 1$ and $l_n(x_n) > 1/2$. Let Y be the closure of the linear span of $\{x_n\}$. Then Y is separable. If $Y \neq X$, then there exists $l \in X^*$ with $\|l\| = 1$ and $l|_Y = 0$. Let us take l_n with $\|l - l_n\| < 1/2$. Then we obtain $|l_n(x_n)| = |(l_n - l)(x_n)| \leqslant \|l_n - l\| < 1/2$, which is a contradiction.

6.10.87.° Let S_n be the operator on $L^1[0, 2\pi]$ that to a function f associates the nth partial sum $S_n(f)$ of its Fourier series. Prove that $\sup_n \|S_n\| = \infty$. Deduce from this that there exist functions in $\mathcal{L}^1[0, 2\pi]$ whose Fourier series do not converge in L^1.

6.10.88.° Let X and Y be Banach spaces, $A \in \mathcal{L}(X, Y)$, and let $\widetilde{A}\colon\ X/\text{Ker}\,A \to Y$, $\widetilde{A}[x] := Ax$, be the operator on the quotient space induced by A.

(i) Prove that $\|\widetilde{A}\| = \|A\|$. (ii) Prove that $\|(\widetilde{A})^* y^*\| = \|A^* y^*\|$, $y^* \in Y^*$.

6.10.89.° Let X and Y be normed spaces and let $J_X : X \to X^{**}$ and $J_Y : Y \to Y^{**}$ be the canonical embeddings. Prove that $T^{**} J_X = J_Y T$ for every operator $T \in \mathcal{L}(X, Y)$. Hence $T^{**} J_X(X) \subset J_Y(Y)$.

6.10.90. Prove that every element $l \in C[0, 1]^*$ is the limit in the weak-$*$ topology of a sequence of functionals of the form $x \mapsto \sum_{i=1}^n c_i x(t_i)$, where $t \in [0, 1]$ and c_i are scalars.

HINT: l is given by a bounded Borel measure μ; one can partition $[0, 1]$ into intervals $I_1 = [0, 1/n)$, $I_2 = [1/n, 2/n)$ and so on and take $t_i = i/n$, $c_i = \mu(I_i)$.

6.10.91. Let C be a convex balanced set in a normed space X and let f be a linear function on X such that the restriction of f to C is continuous at the origin. Prove that f is uniformly continuous on C.

6.10.92. Let X and Y be normed spaces, let X be complete, and let $T \in \mathcal{L}(X, Y)$ be an open mapping. Prove that Y is complete.

6.10.93. (i) Prove that there exists a discontinuous linear mapping that maps a normed space one-to-one onto a Banach space and has a closed graph. (ii) Prove that there exists a discontinuous linear mapping that maps a Banach space one-to-one onto a normed space and has a closed graph.

HINT: (i) consider an injective compact diagonal operator on l^2 and its inverse defined on its range. (ii) Take a Hamel basis $\{h_\alpha\}$ in l^2 with $\|h_\alpha\| = 1$ and consider the norm $\|x\| = \sum_\alpha |x_\alpha|$, $x = \sum_\alpha x_\alpha h_\alpha$ and the identity mapping to this norm from the standard norm on l^2.

6.10.94. Can a continuous linear operator map an incomplete normed space one-to-one onto a complete one?

HINT: use the previous exercise.

6.10.95. Let X and Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. Show that if T takes every bounded closed set to a closed set, then $T(X)$ is closed. Construct an example showing that the closedness of the images of closed balls can be insufficient for this.

HINT: considering the quotient by the kernel reduce the general case to an injective operator and observe that if $Tx_n \to y$, then either $\{x_n\}$ contains a bounded subsequence of $\|x_n\| \to \infty$, so the vectors $x_n/\|x_n\|$ belong to the unit sphere whose image is closed, but $T(x_n/\|x_n\|) \to 0$.

6.10.96. Suppose that a sequence $\{x_n\}$ in a normed space is Cauchy in norm and converges weakly to some vector x. Prove that $\{x_n\}$ converges to x in norm.

HINT: observe that this is true in complete spaces and consider the completion.

6.10.97. Let X be a normed space, $f \in X^*$, $\|f\| = 1$. Prove that f attains its maximum on the unit sphere precisely when $f^{-1}(1)$ has a vector of minimal norm. This is also equivalent to the property that $\mathrm{Ker}\, f$ has a nearest element to some vector outside $\mathrm{Ker}\, f$.

HINT: observe that if e is a vector of minimal norm in $f^{-1}(1)$, then $\|e\| = 1$, since in case $\|e\| > 1$ we can find a unit vector u with $f(u) > 1/\|e\|$ and take $v = u/f(u)$ with $f(v) = 1$ and $\|v\| < \|e\|$. On the other hand, if $f(e) = 1$ and $\|e\| = 1$, then there is no element in $f^{-1}(1)$ with a smaller norm.

6.10.98. (i) Find an example of two closed linear subspaces H_1 and H_2 in a Hilbert space for which $H_1 \cap H_2 = \{0\}$, but the algebraic sum of H_1 and H_2 is not closed (see also Exercise 5.6.57).

(ii)* Prove that such an example exists in every infinite-dimensional Banach space.

HINT: (i) consider the operator T on l^2 given by the formula $Tx = (2^{-n}x_n)$, let H_1 be the graph of T in $l^2 \oplus l^2$, $H_2 := l^2 \oplus \{0\} \subset l^2 \oplus l^2$; then H_1 and H_2 possess the required properties. (ii) Consider a separable infinite-dimensional Banach space X and observe that X^* contains a linearly independent weak-$*$ dense sequence $\{f_n\}$; let L_n be the intersection of the hyperplanes $f_i^{-1}(0)$, $i \leqslant 2n$; take linearly independent vectors x_n and y_n in the algebraic complement of L_n to L_{n+1} with $\|x_n\| \geqslant 1$, $\|y_n\| \geqslant 1$, $\|x_n - y_n\| \leqslant 1/n$ and consider the closed linear spans of $\{x_n\}$ and $\{y_n\}$. To see that their intersection is zero, for each n, find linear combinations φ_n and ψ_n of f_{2n+1} and f_{2n+2} such that $\varphi_n(x_n) = \psi_n(y_n) = 0$, $\varphi_n(y_n) = \psi_n(x_n) = 1$ and observe that the closed linear span of $x_n, x_{n+1}, \ldots$ is contained in the intersection of $\varphi_n^{-1}(0)$ and $f_i^{-1}(0)$ with $i > n$.

6.10.99. Let Y and Z be closed subspaces in a Banach space X and let $Y \cap Z = 0$. Prove that the sum $Y + Z$ is closed precisely when $\mathrm{dist}(S_Y, S_Z) > 0$, where S_Y and S_Z are the unit spheres in Y and Z.

6.10.100. Apply the Hahn–Banach theorem to construct a continuous linear functional l on the set of all bounded functions on the interval $[0,1]$ such that the action of l on continuous functions coincides with the Riemann integral, but there exists a Borel set B for which $l(I_B)$ does not coincide with the Lebesgue measure of B.

6.10.101. Let X and Y be Banach spaces and $A \in \mathcal{L}(X,Y)$. Prove that the equality $A^*(Y^*) = X^*$ holds precisely when A has the zero kernel and a closed range.

HINT: use that A has the zero kernel if and only if the range of A^* is dense and apply Corollary 6.8.6.

6.10.102. Let X and Y be Banach spaces and let X be reflexive. Prove that for every operator $A \in \mathcal{L}(X,Y)$ the image of any closed ball is closed.

HINT: use the weak compactness of balls in X.

6.10.103. Suppose that $C[0,1]$ is equipped with a Banach norm such that convergence in this norm implies the pointwise convergence. Prove that this norm is equivalent to the usual sup-norm.

HINT: use the closed graph theorem.

6.10.104. (Schur's theorem) Prove that in the space l^1 every weakly convergent sequence converges in norm. Deduce from this that l^1 and $L^1[0,1]$ are not linearly homeomorphic.

HINT: consider the case of weak convergence to zero and argue from the opposite.

6.10.105. (i) Prove that l^1 has no subspaces linearly homeomorphic to l^2. (ii) Let $A\colon l^1 \to l^2$ be a continuous linear surjection from Theorem 6.10.8. Prove that l^1 has no subspaces isomorphic to $l^1/\mathrm{Ker}\,A$. In particular, l^1 is not isomorphic to $l^2 \times (l^1/\mathrm{Ker}\,A)$.

HINT: in (i) use Schur's theorem and weak convergence to zero of vectors of the standard basis in l^2; derive (ii) from (i) and the fact that l^2 is isomorphic to $l^1/\mathrm{Ker}\,A$.

6.10.106.° Show that the unit ball in $L^1[0,1]$ has no extreme points (see §5.4).

6.10.107. Prove that the spaces $c_0 \oplus c_0$ (the sum is equipped with the norm equal the sum of the norms of components) and c_0 are isomorphic (i.e., are linearly homeomorphic), but are not isometric.

HINT: if ψ is a linear isometry between these spaces, then each element $\psi(e_n, 0)$, where $\{e_n\}$ is the standard basis in c_0, has 1 and -1 at positions of some finite set S_n. Clearly, $S_n \cap S_m$ if $n \neq k$, since $\|e_n + e_k\| = 1$, $\|e_n - e_k\| = 1$. Hence there is n such that the components of $\psi(0, e_1)$ with indices in S_n are strictly less than 1 in absolute value. This leads to a contradiction, since the norm of (e_n, e_1) equals 2.

6.10.108. Let X and Y be Banach spaces and let $T \in \mathcal{L}(X,Y)$ be an operator such that T^* is an isometry between Y^* and X^*. Is it true that T is an isometry?

6.10.109. Let $f\colon l^\infty \to l^\infty$, $f\colon (x_n) \mapsto \bigl(\tan\bigl((2\pi^{-1}\arctan x_n)^{2n+1}\bigr)\bigr)$. Show that f is continuous and maps l^∞ one-to-one onto l^∞, but f^{-1} is discontinuous at the origin.

6.10.110. Let L be a linear subspace of a normed space X. Prove that every norm on L equivalent to the original one can be extended to a norm on X also equivalent to the original one.

HINT: let U be the unit ball with respect to the original norm and let V be the unit ball in L with respect to the equivalent norm. We can assume that $U \cap L \subset V$. Now take the Minkowski functional of the convex envelope of $U \cup V$.

6.10.111. Show that on the spaces $L^p[0,1]$, $1 \leqslant p < \infty$, there exists no linear lifting, i.e., one cannot choose a representative Lf in every equivalence class $f \in L^p[0,1]$ such that $L(f+g)(t) = Lf(t) + Lg(t)$ and $L(cf) = cf(t)$ for all $f, g \in L^p[0,1]$, all $c \in \mathbb{R}^1$ and all $t \in [0,1]$, and $Lf(t) \geqslant 0$ for all t if $f \geqslant 0$ a.e.

HINT: if L is a linear lifting on $L^p[0,1]$, $1 \leqslant p < \infty$, then for every t the functional $l_t(f) = L(f)(t)$ on $L^p[0,1]$ is linear and nonnegative on nonnegative functions, whence by Proposition 6.10.55 its continuity follows. Hence the functional l_t is given by a function $g_t \in L^{p'}[0,1]$. For every n we partition $[0,1]$ into n intervals $J_{n,1},\ldots,J_{n,n}$ by the points k/n. Let $E_{n,k} := \bigl\{x\colon\ L(I_{J_{n,k}})(x) = 1\bigr\}$ and $E_n := \bigcup_{k=1}^n E_{n,k}$. Then $\lambda(E_n) = 1$ by properties of liftings. There is a point $t \in \bigcap_{n=1}^\infty E_n$. For every n there is $j(n)$ with $t \in E_{n,j(n)}$, i.e., $L(I_{J_{n,j(n)}})(t) = 1$. Since $L(I_{J_{n,k}}) = I_{J_{n,k}}$ a.e., we have

$$L(I_{J_{n,k}})(t) = \int_0^1 I_{J_{n,k}}(s) g_t(s)\, ds \leqslant n^{-1/p} \|g_t\|_{L^{p'}}$$

for all k, which leads to a contradiction.

6.10.112. Extend Goldstein's Theorem 6.7.6 to the case of the sphere: the unit sphere of a Banach space X is dense in the unit sphere of X^{**} with the topology $\sigma(X^{**}, X^*)$.

6.10.113. Prove that every separable Banach space is linearly isometric to a closed subspace in l^∞.

HINT: find a sequence of functionals f_n of unit norm on the given space X such that $\|x\| = \sup_n |f_n(x)|$ for all $x \in X$; observe that it suffices to have the latter for all vectors from a countable dense set.

6.10.114. Prove that on every separable Banach space there is an equivalent strictly convex norm.

HINT: embed this space as a closed subspace in the space $C[0,1]$ with its sup-norm and take the norm $\|x\|_0 = \|x\|_{C[0,1]} + \|x\|_{L^2[0,1]}$.

6.10.115.° Let $\mathcal{K}_n \in C([a,b]^2)$, $n \in \mathbb{N}$, and let K_n be the operator with the integral kernel $\mathcal{K}_n$ on $C[a,b]$. Prove that $K_n f \to f$ for every $f \in C[a,b]$ precisely when 1) $K_n f \to f$ for all f from a dense set, 2) $\displaystyle\sup_n \max_x \int_a^b |\mathcal{K}_n(x,t)|\, dt < \infty$.

6.10.116. Let X be a Banach space and let Y be its n-dimensional subspace. Prove that there exists a linear projection $P\colon\ X \to Y$ for which $\|P\| \leqslant n$.

HINT: use Proposition 6.10.45.

6.10.117. (R. Phillips) Prove that c_0 is not complemented in l^∞.

HINT: see [**9**, §2.5].

6.10.118. Let X be a reflexive Banach space. Prove that every bounded sequence in X contains a weakly convergent subsequence.

6.10.119. Let K be a weakly compact set in a Banach space X such that X^* contains a countable set $\{f_n\}$ separating points of K. Show that $\bigl(K, \sigma(X, X^*)\bigr)$ is a metrizable compact space.

HINT: we can assume that $\|f_n\| \leqslant 1$; take the metric $d(x,y) := \sum_{n=1}^\infty 2^{-n} |f_n(x-y)|$ and verify that the identity mapping from $\bigl(K, \sigma(X, X^*)\bigr)$ to (K, d) is continuous, so it is a homeomorphism.

6.10.120. Let X be a Banach space. Prove that the unit ball in X^* is metrizable in the weak-$*$ topology if and only if X is separable.

HINT: use that if the ball is metrizable in the weak-$*$ topology, then there is a countable collection of vectors in X determining a basis of neighborhoods of zero in the weak-$*$ topology.

6.10.121. Let X be an infinite-dimensional Banach space such that X^* is norm separable. (i) Prove that there exists a sequence of vectors $x_n \in X$ with $\|x_n\| = 1$ weakly converging to zero. (ii) Prove that there exists a sequence of vectors $x_n \in X$ with $\|x_n\| = 1$

weakly converging to zero and a sequence of functionals $l_n \in X^*$ with $l_n(x_n) = 1$ that is weak-$*$ convergent to zero. Consider l^1 to see that (ii) can fail for spaces with nonseparable duals.

HINT: (i) use that X is also separable (Exercise 6.10.86) and the unit ball in X^* is metrizable in the weak-$*$ topology, hence one can pick points in the intersection of the unit sphere with elements of a countable basis of zero in the weak-$*$ topology. (ii) See [**75**, Exercise 3.12.127].

6.10.122.* Let X be a Banach space and let $C \subset X^*$ be a weak-$*$ compact set. (i) Show that if C is norm separable, then the norm closure of the convex envelope of C is also weak-$*$ compact. (ii) Show that without the separability of C the conclusion in (i) can be false by taking for C the set of Dirac measures δ_t in $C[0,1]^*$.

HINT: (i) see [**185**, Exercise 106, p. 104]. (ii) Show that the norm closure of the set of Dirac measures consists of all probability measures concentrated on countable sets, which is not closed in the weak-$*$ topology.

6.10.123. Let X be a metrizable compact set and let $f\colon X \to Y$ be a continuous mapping, where Y is a Hausdorff space. Prove that the compact set $f(X)$ is also metrizable.

HINT: the space $C(X)$ is separable and $C\bigl(f(X)\bigr)$ is isometrically embedded into it by the mapping $\varphi \mapsto \varphi\circ f$. Hence $C\bigl(f(X)\bigr)$ is also separable, which yields the metrizability of $f(X)$.

6.10.124. Let X and Y be Banach spaces and let $A, B \in \mathcal{L}(X,Y)$. Prove that the following conditions are equivalent: (i) $A^*(Y^*) \subset B^*(Y^*)$,

(ii) there is a number k such that $\|Ax\| \leqslant k\|Bx\|$ for all $x \in X$.

HINT: if such k exists, then for every $y^* \in Y^*$ the functional f on $B(X)$ given by the formula $f(Bx) := y^*(Ax)$ is well-defined and bounded, since

$$|y^*(Ax) \leqslant \|y^*\|\,\|Ax\| \leqslant k\|y^*\|\,\|Bx\|.$$

By the Hahn–Banach theorem it extends to an element $z^* \in Y^*$. Then $B^*z^* = A^*y^*$, whence $A^*(Y^*) \subset B^*(Y^*)$. Conversely, if this inclusion holds, then $A^* = \widetilde{B^*}C$, where the operator $\widetilde{B^*}\colon Y^*/\mathrm{Ker}B^* \to X^*$ is generated by the operator B^* and C is a continuous operator (see Theorem 6.10.5). Then $A^{**} = C^*(\widetilde{B^*})^*$, which gives the desired estimate by the known equalities $\|A^{**}\| = \|A^*\| = \|A\|$, $\|(\widetilde{B^*})^*\| = \|\widetilde{B^*}\| = \|B^*\| = \|B\|$ and Exercise 6.10.88.

6.10.125. Let X and Y be Banach spaces and let $A, B \in \mathcal{L}(X,Y)$ be such that $A(X) \subset B(X)$. (i) Prove that there exists a number k such that

$$\|A^*y^*\|_{X^*} \leqslant k\|B^*y^*\|_{X^*}, \quad y^* \in Y^*.$$

(ii) Show that $A^{**}(X^{**}) \subset B^{**}(X^{**})$.

(iii) Show that if X is reflexive, then the estimate in (i) is equivalent to the inclusion $A(X) \subset B(X)$.

HINT: (i) under the same notation as in the hint to the previous exercise we have $A = \widetilde{B}C$, whence

$$\|A^*y^*\|_{X^*} = \|C^*\widetilde{B}^*y^*\|_{X^*} \leqslant \|C^*\|\,\|B^*y^*\|_{X^*}$$

by Exercise 6.10.88. Now (ii) and (iii) follow from the previous exercise.

6.10.126. Let X and Y be Banach spaces, $A \in \mathcal{L}(X,Y)$, $A(X) = Y$, and let Y be separable. Prove that X contains a closed separable subspace Z such that $A(Z) = Y$.

HINT: take a countable set $\{y_n\}$ dense in the unit ball of Y, use Remark 6.2.4 to pick a bounded countable set $\{x_n\} \subset X$ with $Ax_n = y_n$ and take for Z the closure of the linear span of $\{x_n\}$; apply Lemma 6.2.1.

6.10.127.* Prove that the space $C[0,1]$ can be mapped linearly and continuously onto c_0, but not onto l^1. Deduce from this that l^1 embedded isometrically into $C[0,1]$ is not complemented.

HINT: the first assertion is clear from Corollary 6.10.48; for the second assertion, see [**185**, p. 274]. Note that every infinite-dimensional complemented closed subspace in $C[0,1]$ contains a complemented subspace isomorphic to c_0 (see [**185**, Proposition 5.6.4]. There is an unproved conjecture that every infinite-dimensional complemented subspace X in $C[0,1]$ is isomorphic to the space $C(K)$ for some metric compact space K; H. Rosenthal proved that if X^* is nonseparable, then this true with $K = [0,1]$.

6.10.128. (i) Prove that $L^1[0,1]$ contains a complemented subspace isometric to l^1. (ii) Prove that $L^1[0,1]$ embedded isometrically into $C[0,1]$ as a closed subspaces is not complemented.

HINT: (i) take functions constant on $[2^{-1-n}, 2^{-n})$; (ii) apply (i) and the previous exercise. According to an unproved conjecture, every infinite-dimensional complemented subspace X in $L^1[0,1]$ is isomorphic either to l^1 or to $L^1[0,1]$.

6.10.129.* (i) (I. Kaplansky) Let A be a set in a Banach space X and let a point x belong to the closure of A in the weak topology. Prove that x belongs to the weak closure of some countable subset in A.

(ii) Let A be a subset of a weakly compact set in a Banach space X and let a point x belong to the closure of A in the weak topology. Prove that x is the limit of some sequence $\{a_n\} \subset A$ in the weak topology.

HINT: see [**185**, Theorems 4.49 and 4.50, p. 129–130].

6.10.130. (E. A. Lifshits [**695**]) A set W in a Banach space X is called *ideally convex* if the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in X for every bounded sequence $\{x_n\} \subset W$ and every sequence of numbers $\alpha_n \geqslant 0$ with $\sum_{n=1}^{\infty} \alpha_n = 1$.

(i) Prove that in a finite-dimensional space any convex set is ideally convex.

(ii) Give an example of a convex set that is not ideally convex.

(iii) Prove that if a convex set is closed or open, then it is ideally convex.

(iv) Prove that if a set W is ideally convex and $T\colon Z \to X$ is a continuous linear operator from a Banach space Z, then $T^{-1}(W)$ is ideally convex.

(v) Prove that if a set W is ideally convex and bounded and $T\colon X \to Y$ is a continuous linear operator to a Banach space Y, then $T(W)$ is ideally convex.

(vi) Let W be an ideally convex set. Prove that the interior of W coincides with the interior of the closure of W and also with the algebraic kernel of W and the algebraic kernel of the closure of W.

(vii) From the previous results deduce the Banach–Steinhaus theorem and the open mapping theorem.

6.10.131. Let E be a closed linear subspace in $C[0,1]$ such that $E \subset C^1[0,1]$. Prove that $\dim E < \infty$.

HINT: observe that $C^1[0,1]$ belongs to the range of a compact operator and hence cannot contain infinite-dimensional closed subspaces.

6.10.132.° Let X and Y be normed spaces and let $A\colon X \to Y$ be a linear mapping continuous from the weak topology to the norm topology. Prove that $A(X)$ is finite-dimensional.

HINT: use that a weak neighborhood of zero in an infinite-dimensional space contains a subspace of finite codimension.

6.10.133. Let X and Y be Banach spaces and let $J_X: X \to X^{**}$ and $J_Y: Y \to Y^{**}$ be the canonical embeddings. Let $S \in \mathcal{L}(Y^*, X^*)$ and $S^* J_X(X) \subset J_Y(Y)$. Prove that $S = T^*$, where $T \in \mathcal{L}(X, Y)$ is defined as follows: $T = J_Y^{-1} S^* J_X$.

6.10.134. Let X be a normed space, $M > 0$, $\{x_n\} \subset X$, $\{c_n\} \subset \mathbb{R}^1$. Prove that the existence of a functional $f \in X^*$ with $\|f\| \leqslant M$ and $f(x_n) = c_n$ for all n is equivalent to the condition that $\bigl|\sum_{i=1}^n \lambda_i c_i\bigr| \leqslant M \bigl\|\sum_{i=1}^n \lambda_i x_i\bigr\|$ for all n and all $\lambda_i \in \mathbb{R}^1$.

6.10.135° Suppose that we are given two sequences of numbers $a_n > 0$ and $b_n > 0$, where $\{b_n\}$ decreases to zero and $\{a_n b_n\}$ has a finite limit. Let us define an operator $T\colon C[0,1] \to c$ by the formula

$$(Tx)_n := a_n \int_0^{b_n} x(t)\, dt.$$

Prove that K is compact precisely when $a_n b_n \to 0$.

HINT: show that $\|K\| = \sup_n |a_n b_n|$.

6.10.136. (Holmgren's theorem) Let μ and ν be probability measures on spaces Ω_1 and let Ω_2 and let $\mathcal{K}$ be a $\mu\otimes\nu$-measurable function such that

$$C_1 = \operatorname{ess\,sup}_s \int_{\Omega_2} |\mathcal{K}(s,t)|\, \nu(dt) < \infty, \quad C_2 = \operatorname{ess\,sup}_t \int_{\Omega_1} |\mathcal{K}(s,t)|\, \mu(ds) < \infty.$$

Prove that the operator

$$Kx(t) = \int_{\Omega_2} \mathcal{K}(s,t) x(s)\, \mu(ds), \quad K\colon L^2(\mu) \to L^2(\nu),$$

is bounded and $\|K\| \leqslant C_1^{1/2} C_2^{1/2}$.

HINT: let $C_2 > 0$ and $c = C_1^{1/2} C_2^{-1/2}$; for $x \in \mathcal{L}^\infty(\mu)$ and $y \in \mathcal{L}^\infty(\nu)$ such that $\|x\|_{L^2(\mu)} \leqslant 1$ and $\|y\|_{L^2(\nu)} \leqslant 1$, we have

$$\biggl|\int_{\Omega_2}\int_{\Omega_1} \mathcal{K}(s,t) x(s) y(t)\, \mu(ds)\, \nu(dt)\biggr| \leqslant \int_{\Omega_2}\int_{\Omega_1} |\mathcal{K}(s,t)| \bigl[2^{-1} c |y(t)|^2 + 2^{-1} c^{-1} |x(s)|^2\bigr]\, \mu(ds)\, \nu(dt) \leqslant 2^{-1} c C_2 + 2^{-1} c^{-1} C_1 = C_1^{1/2} C_2^{1/2},$$

whence the desired bound follows.

6.10.137. (Schur's test) Let μ and ν be nonnegative measures on measurable spaces T and S, $\mathcal{K} \geqslant 0$ a measurable function on $T\times S$, and let $\varphi > 0$ and $\psi > 0$ be measurable functions on T and S, respectively, such that

$$\int_S \mathcal{K}(t,s)\psi(s)\, \nu(ds) \leqslant \alpha\varphi(t) \quad \mu\text{-a.e.},$$

$$\int_T \mathcal{K}(t,s)\varphi(t)\, \mu(dt) \leqslant \beta\psi(s) \quad \nu\text{-a.e.},$$

where α and β are numbers. Prove that the operator

$$K\colon L^2(\nu) \to L^2(\mu), \quad Kx(t) := \int_S \mathcal{K}(t,s) x(s)\, \nu(ds),$$

is bounded and $\|K\|^2 \leqslant \alpha\beta$. Deduce from this the assertion of the previous exercise.

HINT: observe that

$$\|Kx\|^2_{L^2(\mu)} \leqslant \int_T \biggl(\int_S \mathcal{K}(t,s)\psi(s)\,\nu(ds)\biggr)\biggl(\int_S \mathcal{K}(t,s)\psi(s)^{-1}|x(s)|^2\,\nu(ds)\biggr)\,\mu(dt)$$
$$\leqslant \alpha \int_T \varphi(t)\biggl(\int_S \mathcal{K}(t,s)\psi(s)^{-1}|x(s)|^2\,\nu(ds)\biggr)\,\mu(dt) \leqslant \alpha\beta\|x\|^2_{L^2(\nu)}.$$

6.10.138. Prove that the formula

$$Ax(t) = \frac{1}{t}\int_0^t x(s)\,ds$$

defines a bounded operator on $L^2[0,1]$.

HINT: apply Schur's test to the kernel $\mathcal{K}(t,s) = t^{-1}$ if $s < t$ and $\mathcal{K}(t,s) = 0$ if $s \geqslant t$ and the function $\varphi(t) = \psi(t) = t^{-1/2}$.

6.10.139. Let X and Y be Banach spaces and $A \in \mathcal{L}(X,Y)$. Prove that the compactness of the operator A is equivalent to the following condition: there exist functionals $l_n \in Y^*$ with $\|l_n\| \to 0$ such that $\|Ax\| \leqslant \sup_n |l_n(x)|$ for all $x \in X$.

Deduce from this assertion that the compactness of A is equivalent to the existence of a bounded sequence $\{f_n\}$ in X^* and a sequence of numbers λ_n such that $\lambda_n \to 0$ and $\|Ax\| \leqslant \sup_n |\lambda_n f_n(x)|$ for all $x \in X$.

HINT: if $A \in \mathcal{K}(X,Y)$, then $A^* \in \mathcal{K}(Y^*,X^*)$. Let W be the unit ball in Y^*. Then $A^*(W)$ is contained in the closure of the convex envelope of the sequence $l_n \to 0$ (Proposition 5.5.7). Hence for every $x \in X$ we have

$$\|Ax\| = \sup_{f\in W} |f(Ax)| = \sup_{l\in A^*(W)} |l(x)| \leqslant \sup_n |l_n(x)|.$$

Conversely, if the indicated condition is fulfilled, then $S = \{l_n\} \cup \{0\}$ is compact in Y^*. The image of the unit ball U in X is totally bounded as a set in $C(S)$ by the Arzelà–Ascoli theorem, since $|l(Ax) - l'(Ax)| \leqslant \|A\|\,\|l - l'\|$ whenever $\|x\| \leqslant 1$, which means the uniform Lipschitzness of the elements Ax as functions on S. The estimate from our condition shows that $\|Ax\|$ is estimated by the norm of Ax as an element of $C(S)$. Hence the set $A(U)$ is totally bounded in Y. The second assertion follows from the first one.

6.10.140. Let X and Y be Banach spaces and let Y be separable. Prove that the compactness of an operator $A \in \mathcal{L}(X,Y)$ is equivalent to the following condition: for every sequence $\{y_n^*\} \subset Y^*$ that is weak-$*$ convergent to zero we have $\|A^*y_n^*\| \to 0$.

HINT: in one direction, use that the adjoint of a compact operator is compact. In the opposite direction, use that for a bounded sequence of vectors $x_n \in X$, the sequence of vectors $Ax_n \in Y$ regarded as functions on the unit ball in Y^* with a metric defining the weak-$*$ topology is uniformly equicontinuous due to the given condition.

6.10.141. Let X be a reflexive Banach space. Prove that every operator $A \in \mathcal{L}(X, l^1)$ is compact.

HINT: use that weakly convergent sequences in l^1 are norm convergent.

6.10.142. (i) Let μ be a nonnegative measure on a measurable space $(\Omega, \mathcal{A})$ and let $(t,s) \mapsto \mathcal{K}(t,s)$ be a measurable function such that the function $s \mapsto \mathcal{K}(t,s)$ belongs to $\mathcal{L}^2(\mu)$ for μ-a.e. t and the function

$$Kx(t) = \int_\Omega \mathcal{K}(t,s)x(s)\,\mu(ds)$$

belongs to $\mathcal{L}^2(\mu)$ for all $x \in \mathcal{L}^2(\mu)$. Prove that K is a bounded operator on $L^2(\mu)$.

(ii) Generalize (i) to the case where it is only known that, for every function x from $\mathcal{L}^2(\mu)$, the function $s \mapsto \mathcal{K}(t,s)x(s)$ is integrable and $Kx \in \mathcal{L}^2(\mu)$.

(iii) Show that if in (i) or (ii) the operator K is zero, then $\mathcal{K}(t,s) = 0$ $\mu\otimes\mu$-a.e.

(iv) Let μ be Lebesgue measure on $[0,1]$. Prove that the unit operator I cannot be represented in the form indicated in (ii).

(v) Let the measure μ in (ii) be finite. Show that the operator $K\colon L^2(\mu) \to L^1(\mu)$ is compact, although it need not be compact as an operator with values in $L^2(\mu)$.

(vi) Let μ be Lebesgue measure on $[0,1]$, $1/2 \leqslant \alpha < 1$, and let $\mathcal{K}_\alpha(t,s) = |t-s|^{-\alpha}$ if $t > s$ and $\mathcal{K}_\alpha(t,s) = 0$ if $t \leqslant s$. Prove that the integral kernel $\mathcal{K}_\alpha$ generates a bounded operator on $L^2[0,1]$.

HINT: in (i) and (ii) apply the closed graph theorem (see [**252**, Theorem 3.10]). For assertions (iii)–(vi), see [**252**, Theorem 8.1, Theorem 8.5, Theorem 13.8, Example 11.1].

6.10.143. (i) Let $\Omega \subset \mathbb{R}^d$ be a bounded measurable set, let $\mathcal{K}_0$ be a bounded measurable function, and let $\mathcal{K}(t,s) = \mathcal{K}_0(t,s)|t-s|^{-\alpha}$, where $\alpha < d$. Prove that the operator K defined by the kernel $\mathcal{K}$ on $L^2(\Omega)$ is compact.

(ii) Prove that if the function $\mathcal{K}_0$ is continuous in t, then the operator K is compact also on $C(\Omega)$.

6.10.144.° Is the operator $Ax(t) = x(\sqrt{t})$ compact on $C[0,1]$? On $L^2[0,1]$?

6.10.145.° Suppose we are given a sequence of disjoint intervals $[a_n, b_n]$ in $[0,1]$. Prove that the operator on $L^1[0,1]$ defined by the integral kernel

$$\mathcal{K}(t,s) = \sum_{n=1}^{\infty} (b_n - a_n)^{-1} I_{[a_n,b_n]}(t) I_{[a_n,b_n]}(s),$$

is not compact.

6.10.146. Prove that every compact operator T on $L^1[0,1]$ can be expressed in the form $Tx(t) = \displaystyle\int_0^1 \mathcal{K}(t,s)x(s)\,ds$ with a measurable kertnel $\mathcal{K}(\,\cdot\,,\,\cdot\,)$ such that the condition $\sup_s \|\mathcal{K}(\,\cdot\,,s)\|_{L^1} < \infty$ is fulfilled.

HINT: see [**164**, p. 508].

6.10.147. Let X and Y be Banach spaces. An operator $T\colon X \to Y$ is called *completely continuous* if it takes weakly compact sets to norm compact sets.

(i) Prove that an operator $T \in \mathcal{L}(X,Y)$ is completely continuous precisely when it takes weakly convergent sequences to norm convergent sequences.

(ii) Prove that the set of completely continuous operators is a closed linear subspace in $\mathcal{L}(X,Y)$.

(iii) Prove that every operator $T \in \mathcal{L}(l^1, Y)$ is completely continuous. In particular, the identity operator on l^1 is completely continuous, but not compact.

6.10.148. Let X be an infinite-dimensional Banach space. Prove that X^* with the weak-$*$ topology is not metrizable. Show that the completeness is important considering the subspace E of finite sequences in c_0 (here $E^* = l^1$, but $\sigma(l^1,E) \neq \sigma(l^1,c_0)$).

HINT: observe that if such a metric exists, then the corresponding ball of radius $1/n$ centered at zero in X^* must contain a weak-$*$ neighborhood of zero, hence must contain a functional f_n with $\|f_n\| = n$, which is impossible, since the sequence $\{f_n\}$ converges to zero pointwise, but is not norm bounded.

6.10.149. Give an example of a sequence of continuous linear functionals f_n on a Banach space X that is weak-$*$ convergent to zero, but the convex envelope of $\{f_n\}$ is contained in the sphere $\{f\colon \|f\| = 1\}$.

HINT: consider $X = c_0$ and coordinate functions; use that $c_0^* = l^1$ and that convex combinations of coordinate functions have unit norms in l^1.

6.10.150° Let H be a Hilbert space and let $A_n \in \mathcal{L}(H)$ be such that $A_n x \to 0$ for every $x \in H$. Is it true that $A_n^* x \to 0$ for every x?

6.10.151. Let X and Y be normed spaces and let $S\colon\ Y^* \to X^*$ be a linear mapping. Prove that the existence of an operator $T \in \mathcal{L}(X, Y)$ for which $S = T^*$ is equivalent to the continuity of S with respect to the topologies $\sigma(Y^*, Y)$ and $\sigma(X^*, X)$. In particular, the continuity of S with respect to the weak-$*$ topologies yields the norm continuity of S.

HINT: if the operator S is continuous with respect to the weak-$*$ topologies, then, for every $x \in X$, the functional $y^* \mapsto Sy^*(x)$ on Y^* is continuous in the topology $\sigma(Y^*, Y)$, hence there exists an element $Tx \in X$ for which $Sy^*(x) = y^*(Tx)$; verify that T is the required operator.

6.10.152. Let $X = c_0$. The formula $Sy = \bigl(\sum_{n=1}^{\infty} y_n, y_2, y_3, \ldots\bigr)$, $y = (y_n)$, defines a bounded operator on $X^* = l^1$. Prove that S maps X^* one-to-one onto X^*, i.e., is a linear homeomorphism of X^*, but cannot be the adjoint for an operator $T \in \mathcal{L}(X)$.

HINT: one has the weak-$*$ convergence $e_n \to 0$, where $\{e_n\}$ is the standard basis in l^1, but $(Se_n)_1 = 1$ for all n, i.e., S is not continuous in the weak-$*$ topology.

6.10.153. Let X and Y be Banach spaces and let Y be separable. Let $S \in \mathcal{L}(Y^*, X^*)$. Prove that the existence of an operator $T \in \mathcal{L}(X, Y)$ such that $T^* = S$ is equivalent to the following condition: if $y_n^* \to 0$ in the weak-$*$ topology Y^*, then $Sy_n^* \to 0$ in the weak-$*$ topology X^*.

6.10.154. Prove Theorem 6.10.44 by an inductive construction without using the embedding into $C[0, 1]$ in the following stronger formulation: if we are given two sequences $\{y_n\} \subset X$ and $\{f_n\} \subset X^*$ and the latter separates points in X, then a Markushevich basis $\{x_n\}$ in the linear span of $\{y_n\}$ can be chosen in such a way that the corresponding sequence $\{l_n\} \subset X^*$ exists in the linear span of $\{f_n\}$.

HINT: see [**185**, p. 188].

6.10.155. Let X be an infinite-dimensional separable Banach space. Show that there exist sequences $\{x_n\} \subset X$ and $\{l_n\} \subset X^*$ such that $l_i(x_j) = \delta_{ij}$, the linear span of $\{x_n\}$ is dense in X, but there exists a nonzero element $x \in X$ with $l_n(x) = 0$ for all n.

HINT: take a sequence of linearly independent vectors a_i with $\|a_i\| = 1$ whose linear span L is dense in X; take $x \in X \backslash L$. By the Hahn–Banach theorem there exist $f_i \in X^*$ with $\|f_i\| = 1$, $f_i(x) = 0$, $f_i(a_i) = 1$ and $f_i(a_j) = 0$ for $j = 1, \ldots, i-1$. One can find vectors $x_n \in L$ and functionals l_n in the linear span of $\{f_i\}$ with $l_i(x_j) = \delta_{ij}$ (see Exercise 6.10.154); one has $l_n(x) = 0$, since $f_i(x) = 0$ for all i.

6.10.156. Let X be a real normed space with the closed unit ball U and let $f, g \in X^*$ be such that $\|f\| = \|g\| = 1$ and $f^{-1}(0) \cap U \subset g^{-1}([-\varepsilon, \varepsilon])$, where $0 < \varepsilon < 1/2$. Prove that either $\|f - g\| \leqslant 2\varepsilon$ or $\|f + g\| \leqslant 2\varepsilon$.

HINT: see [**77**, p. 128].

6.10.157. Prove that the functions $\varphi_n = (n + 1/\pi)^{1/2} z^n$ form an orthonormal basis in the Bergman space $A^2(U)$ (see Example 5.2.2), where U is the unit disc in $\mathbb{C}^1$.

HINT: verify that these functions are mutually orthogonal and that for every function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ from $A^2(U)$ we have Parseval's relation $\|f\|_{L^2}^2 = \sum_{n=0}^{\infty} |(f, \varphi_n)|^2$. To this end, it suffices to show that $\|f\|_{L^2}^2 = \pi \sum_{n=0}^{\infty} |c_n|^2/(n+1)$, which is verified by evaluating the integral of $|f(z)|^2$ over the disc of radius $r < 1$, where the series converges uniformly.

6.10.158. Let $\mathcal{A}(U)$ be the space of analytic functions on the open unit disc U in $\mathbb{C}^1$ that are continuous on the closure of U. Equip $\mathcal{A}(U)$ with the norm $\|\varphi\| = \max_z |\varphi(z)|$. Prove that any continuous linear functional on $\mathcal{A}(U)$ has the form

$$l(\varphi) = \lim_{q\to 1-} \sum_{k=0}^{\infty} q^k \frac{\varphi^{(k)}(0)}{k!} l(z^k), \quad \sup_k |l(z^k)| < \infty.$$

6.10.159. Let V be a nonempty convex closed set in a reflexive Banach space X. Prove that every point $x \in X$ has a nearest point in V.

6.10.160. Let X be a Banach space. Prove that the following conditions are equivalent: (i) X is reflexive, (ii) every nonempty closed convex set in X has a point nearest to the origin, (iii) for every closed separable linear subspace $Y \subset X$ and every functional $f \in Y^*$ with $\|f\| = 1$ the set $f^{-1}(1)$ contains a point nearest to the origin.

HINT: use Theorem 6.10.10 and Exercise 6.10.97.

6.10.161. Let X be a Banach space that is not reflexive. Prove that X^* contains a norm closed linear subspace that is not closed in the weak-$*$ topology.

HINT: take an element $F \in X^{**}$ not belonging to the image of X under the canonical embedding $X \subset X^{**}$ and consider the kernel of F.

6.10.162. Prove that every Efimov–Stechkin normed space (see Exercise 5.6.66) is complete and reflexive.

6.10.163. Prove that for any infinite-dimensional separable Banach spaces X_1 and X_2 there is a compact operator $T\colon\ X_1 \to X_2$ with the zero kernel and a dense range containing an a priori given sequence from X_2.

HINT: if $X_2 = l^2$, then one can take a sequence $\{l_n\} \subset X_1^*$ separating points in X_1 with $\|l_n\| \leqslant 1$ and set $(Tx)_n = 2^{-n} l_n(x)$; if $X_1 = l^2$, then one can take a sequence of unit vectors $y_n \in X_2$ with a dense linear span, set $T_0 x = \sum_{n=1}^{\infty} 2^{-n} x_n y_n$ and take the operator $T_0/\operatorname{Ker} T_0$.

6.10.164. (i) Let E_1 and E_2 be Hilbert spaces and let $A \in \mathcal{L}(E_1, E_2)$. Suppose that E_2 is separable and the operator A is injective. Prove that E_1 is also separable.

(ii) Extend (i) to the case where E_1 and E_2 are Banach spaces and E_1 is reflexive. Give an example showing that this can be false if E_1 is not reflexive.

(iii) Show that if in (i) we do not assume the separability of E_2, then one can assert that the cardinality of an orthonormal basis of the space E_1 does not exceed the cardinality of an orthonormal basis in the space E_2 and that there exists an operator $B \in \mathcal{L}(E_2)$ such that $B(E_2) = E_1$.

HINT: (i) the set $A^*(E_2^*)$ is dense in E_1^* by the injectivity of A; the separability of E_1^* implies the separability of E_1. (ii) Embed E_2 injectively into $L^2[0,1]$. (iii) Use the density of $A^*(E_2^*)$ in E_1^* and Exercise 5.6.53.

6.10.165. Let H be a separable Hilbert space. (i) Give an example of bounded operators A and B on H for which the sets $A(H)$ and $B(H)$ are dense in H, but $A(H) \cap B(H) = 0$.

(ii) Let $A \in \mathcal{L}(H)$ and let $A(H)$ be non-closed. Prove that there exists an operator $B \in \mathcal{L}(H)$ such that $B(H)$ is dense and $A(H) \cap B(H) = 0$.

HINT: see Theorem 7.10.18 in the next chapter.

6.10.166. Let X be a Banach space. A sequence of vectors $x_n \in X$ is called ω-independent if the relation $\sum_{n=1}^{\infty} c_n x_n = 0$ implies that $c_n = 0$ for all n.

(i) Give an example of a linearly independent sequence that is not ω-independent.

(ii) (V. I. Gurarii) Let $\{x_n\} \subset X$ and a nonzero $x_0 \in X$ be such that for some $C > 0$ and all n one has

$$\sum_{k=n+1}^{\infty} \|x_k - x_0\| < C\varrho_n, \quad \varrho_n := \inf\Bigl\{\Bigl\|x_0 - \sum_{k=1}^{n} \alpha_k x_k\Bigr\|\colon\ \max_{k\leqslant n} |\alpha_k| \geqslant 1\Bigr\}.$$

Prove that $\{x_n\}$ is ω-independent.

(iii) Prove that every linearly independent sequence contains an infinite ω-independent subsequence.

HINT: see [**690**].

6.10.167. (i) Prove that the norm on any infinite-dimensional normed space is not continuous in the weak topology.

(ii) Prove that the norm on the space l^1 is sequentially continuous in the weak topology, although is not continuous.

(iii) Prove that the function $f(x) = \sum_{n=1}^{\infty} n^{-2} x_n^2$ on l^2 with the weak topology is sequentially continuous, but is discontinuous at every point.

HINT: (i) observe that the norm is not bounded on weak neighborhoods of zero; (ii) apply Schur's theorem from Exercise 6.10.104; (iii) use the compactness of the operator $A\colon\ (x_n) \mapsto (n^{-1}x_n)$ to verify the weak sequential continuity (or use the uniform convergence of the series on balls); to prove the discontinuity of f in the weak topology show that the operator A cannot be bounded on a weak neighborhood of zero.

6.10.168. Let X be a Banach space. Prove that any two closed subspaces in X of codimension 1 are linearly homeomorphic. Deduce from this that any two closed subspaces in X of the same finite codimension are linearly homeomorphic.

HINT: see [**185**, Exercise 2.7, p. 53].

6.10.169. Prove that a Banach space X is linearly homeomorphic to $X \oplus \mathbb{R}^1$ precisely when X is linearly homeomorphic to every closed hyperplane in X.

HINT: use the previous exercise and the fact that the space X is linearly homeomorphic to $H \oplus \mathbb{R}^1$, where H is a closed hyperplane in X. Note that there exists an infinite-dimensional separable Banach space X that cannot be linearly homeomorphic to its closed hyperplane (see [**689**]).

6.10.170.* Prove that every closed hyperplane in $C[0,1]$ is linearly homeomorphic to the whole space $C[0,1]$.

HINT: see [**185**, Exercise 5.33, p. 153].

6.10.171.* Let l be a discontinuous linear function on a Banach space. (i) Can $l^{-1}(0)$ be a second category set?

(ii)** Can $l^{-1}(0)$ be a first category set? (See [**667**].)

6.10.172. Prove that every bounded closed convex set in a Hilbert space is the intersection of some family of closed balls.

HINT: observe that every point in the complement of this set is outside some ball containing this set. A more general result is mentioned in Theorem 6.10.36.

6.10.173. (i) Show that the closure of the convex envelope of an orthonormal basis in l^2 has no interior points. (ii)* Show that the closure of the convex envelope of a weakly convergent sequence in an infinite-dimensional Banach space has no interior points.

HINT: (ii) see [**185**, p. 87].

6.10.174.* Prove that there exists a continuous linear surjection $T\colon\ C[0,1] \to L^2[0,1]$.

HINT: see [**185**, p. 195] or [**75**, Exercise 3.12.188, p. 241].

6.10.175. Let K be a compact space. Show that the extreme points (see §4.6) of the unit ball of $C(K)$ are the functions with values in $\{1,-1\}$, and the extreme points of the unit ball of $C(K)^*$ are Dirac's measures δ_k and the measures $-\delta_k$, $k \in K$.

6.10.176.* Prove Theorem 5.6.5: if K_1 and K_2 are compact spaces, then $C(K_1)$ and $C(K_2)$ are linearly isometric precisely when K_1 and K_2 are homeomorphic.

HINT: if $h\colon K_1 \to K_2$ is a homeomorphism, then $J(f) := f \circ h$ is a linear isometry between $C(K_2)$ and $C(K_1)$. Conversely, if $J\colon C(K_1) \to C(K_2)$ is a linear isometry, then $J^*\colon C(K_2)^* \to C(K_2)^*$ is an isometry. For every $k \in K_2$, the measure $J^*\delta_k$ is a point of the unit ball in $C(K_1)^*$. By Exercise 6.10.175 one has $J^*\delta_k = \varepsilon_k \delta_{h(k)}$, where $h(k) \in K_1$, $\varepsilon_k = 1$ or $\varepsilon_k = -1$. It is readily seen that the mapping $k \mapsto \varepsilon_k \delta_{h(k)}$ is continuous, since J^* is continuous with respect to the weak-$*$ topologies. Moreover, the function $k \mapsto \varepsilon_k$ is also continuous, because $\varepsilon_k = \varepsilon_k \delta_{h(k)}(1) = J^*\delta_k(1) = J(1)(k)$. Therefore, the mapping $h\colon k \mapsto h(k)$ is continuous. It gives the required homeomorphism. About recovering of K by $C(K)$ see [**304**, § 18.2.1].

6.10.177. (The Alekhno–Zabreiko theorem) Suppose that functions $f_n \in L^\infty[0,1]$ converge to zero in the topology $\sigma\bigl(L^\infty, (L^\infty)^*\bigr)$. Prove that $f_n(t) \to 0$ a.e.

HINT: if Λ is a lifting on $L^\infty[0,1]$, then $\Lambda(f_n)(t) \to 0$ for all t.

6.10.178. Prove that $L^1[0,1]$ possesses the Dunford–Pettis property.

HINT: apply Exercise 6.10.177.

6.10.179. Let X be a Banach space. Prove that X^* is complemented in X^{***}.

HINT: set $P\colon X^{***} \to X^*$, $P(f) := f|_{X^*}$ (the Dixmier projection).

6.10.180. (*The James space*) Let J be the linear subspace in c_0 consisting of all elements with finite norm

$$\|x\|_J := \sup\Bigl((x_{j_2} - x_{j_1})^2 + \cdots + (x_{j_{2m}} - x_{j_{2m-1}})^2 + (x_{j_{2m+1}})^2\Bigr)^{1/2},$$

where sup is taken over all finite collections $1 \leqslant j_1 < j_2 < \cdots < j_{2m+1}$. Prove that the space J has codimension 1 under the canonical embedding into J^{**} and hence is not reflexive, however, it is linearly isometric to J^{**}. Deduce from this that J cannot be isomorphic to $X \oplus X$ for a Banach space X. In particular, J is not isomorphic to $J \oplus J$.

6.10.181. Let $\{h_n\}$ be a Schauder basis in a Hilbert space H and let $\{l_n\}$ be a sequence of functionals on H such that $l_i(h_j) = \delta_{ij}$ and $\|l_n\| = \|h_n\| = 1$. Prove that $\{h_n\}$ is an orthonormal basis.

HINT: if $(h_1, h_2) \neq 0$, then the linear span of h_1 and h_2 contains a unit vector $v \perp h_1$. Then $h_2 = (h_2, v)v + (h_2, h_1)h_1$, whence we obtain $1 = |(h_2, v)|^2 + |(h_2, h_1)|^2$ and $|(h_2, v)| < 1$. However, $1 = |l_2(h_2)| = |(h_2, v)|\,|l_2(v)|$, which gives $|l_2(v)| > 1$. Hence $\|l_2\| > 1$, a contradiction.

6.10.182.° Let $\{e_n\}$ be a Schauder basis in a Banach space X. Prove that $K \subset X$ is totally bounded precisely when for each $\varepsilon > 0$ there is n_ε with $\sup_{x\in K} \bigl\|\sum_{k=n_\varepsilon}^\infty c_k(x) e_k\bigr\| \leqslant \varepsilon$, where $x = \sum_{k=1}^\infty c_k(x) e_k$.

6.10.183.* Let X be a Banach space with a Schauder basis $\{h_n\}$ and let $\{l_n\}$ be the corresponding coordinate functionals. Prove that $\{l_n\}$ is a Schauder basis in X^* precisely when the linear span of $\{l_n\}$ is dense in X^*.

HINT: see [**417**, p. 405].

6.10.184. Let H be a Hilbert space, $A \in \mathcal{L}(H)$ and $\|A\| \leqslant 1$. Prove that the operators $S_n := n^{-1}(I + A + \cdots + A^{n-1})$ converge pointwise to the projection onto $\operatorname{Ker}(A - I)$. For more general results, see [**354**].

6.10.185. Let H_1 and H_2 be mutually orthogonal infinite-dimensional closed subspaces in a separable Hilbert space. Show that there is an infinite-dimensional closed subspace H_3 such that $H_1 \cap H_3 = H_2 \cap H_3 = 0$.

6.10.186. (Hardy's inequality) Let $p \in (1, +\infty)$. Prove that the operator

$$A\colon (x_n) \mapsto \Bigl(x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + x_2 + \cdots + x_n}{n}, \ldots\Bigr)$$

is bounded on l^p and its norm is $p/(p-1)$.

HINT: see [**254**, §9.8].

6.10.187. (i) Let H be an infinite-dimensional Hilbert space. Consider the mapping $(A, B) \mapsto AB$, $\mathcal{L}(H)\times\mathcal{L}(H) \to \mathcal{L}(H)$. Investigate the continuity and sequential continuity of this mapping equipping $\mathcal{L}(H)$ with one of the following operator topologies: (a) the norm topology, (b) the strong operator topology, (c) the weak operator topology. Consider also different combinations of topologies on the factors and on the range.

(ii) Show that if H is separable, then on norm bounded sets in $\mathcal{L}(H)$ the weak and strong operator topologies are metrizable.

HINT: investigating the sequential continuity use the fact that a pointwise convergent sequence of operators is norm bounded. To disprove the continuity of multiplication in the weak operator topology use the left and right shifts L and R on the space $l^2(\mathbb{Z})$ of two-sided sequence, for which $L^n \to 0$ and $R^n \to 0$ in the weak operator topology, but $L^n R^n = I$. To disprove the continuity of multiplication in the strong operator topology observe that, given vectors $v_1, \ldots, v_n$ and u and a number $\varepsilon > 0$, one can find a bounded operator A such that $\|Av_i\| < \varepsilon$, $i = 1, \ldots, n$, but $\|A^2 u\| > 1$.

6.10.188. Let A be a closed set in a Hilbert space contained in the open unit ball. Prove that the closure of A in the weak topology is also contained in the open unit ball.

HINT: show that every point in the unit sphere can be separated from A by a closed hyperplane.

6.10.189. Show that there is no sequence of closed subsets F_n in l^2 contained in the open unit ball and having the property that every closed set in l^2 contained in the open unit ball belongs to some F_n.

HINT: take the standard orthonormal basis $\{e_n\}$ and observe that for any numbers $\varepsilon_n \in (0, 1/2)$ the set of vectors $(1 - \varepsilon_n)e_n$ is closed.

6.10.190.* Let X be a Banach space and let $L \subset X^*$ be a linear subspace separating points in X. Is it true that for every $x \in X$ one has $\|x\| = \sup\{l(x)\colon l \in L,\ \|l\| \leqslant 1\}$? Consider $X = c_0$, $L = \{y = (y_n) \in l^1\colon y_k = k^{-1} \sum_{n \in I_k} y_n \quad \forall k \in I\}$, where $I, I_1, I_2, \ldots$ are infinite disjoint sets whose union is $\mathbb{N}$.

6.10.191.* Construct an example of a Banach space X and a linear subspace $L \subset X^*$ with the following property: L is dense in X^* in the topology $\sigma(X^*, X)$, but the weak-$*$ closure of the intersection of L with the unit ball of X^* contains no ball of a positive radius.

HINT: see [**85**, p. 275].

6.10.192.* (i) Prove that there exists a linear mapping $A\colon l^2 \to l^1$ discontinuous on every infinite-dimensional linear subspace in l^2 (not necessarily closed).

(ii) Prove that for every linear mapping A from l^1 to a Banach space E, there exists an infinite-dimensional linear subspace L in l^1 such that the restriction of A to L is continuous.

(iii) Prove that for every linear mapping $A\colon l^2 \to l^2$, there exists an infinite-dimensional linear subspace L in l^2 such that the restriction of the mapping A to L is continuous. For more general facts, see [**676**], [**674**], and [**706**].